NUMERICAL INVESTIGATIONS OF THE KORTEWEG-DE VRIES (KdV) EQUATION

BRIAN R. JOHANSEN

Numerical Investigations of the Korteweg-de Vries (KdV) Equation

by

A thesis submitted to the

School of Graduate Studies

in partial fulfilment of the

requirements for the degree of

Master of Science

Computational Science

Memorial University of Newfoundland

June 2008

St. John's Newfoundland

Abstract

This thesis will develop material regarding the Korteweg-de Vries (KdV) equation, a nonlinear partial differential equation which has soliton solutions. We introduce the equation with its history and establish some preliminaries in §1. In §2, we will examine the soliton solutions and the uniqueness of such. We will also speak of the construction of multiple soliton solutions, as well as other solutions. Next, the conservation properties of the KdV equation will be visited, then the properties of interacting solitons. In §3 we will discuss the historical numerical schemes for the KdV equation, including finite difference methods, pseudospectral methods, collocation, and finite element methods. We will comment on their accuracy and efficiency. Contained within §4 is a selection of numerical schemes which were implemented (and in one case, improved!) by the author.

Acknowledgements

At this official benchmark of the author's goal to approach omniscience in all selected areas of meaningful knowledge, he would like to express somewhat lengthy (and perhaps verbose) acknowledgements to persons (and entities) of special note who have influenced this body of work to a significant degree. Those referenced here are some members of the entire set of those to whom the author would like to express gratitude, that is, this list is not exclusive.

Most importantly, the contents herein would never have commenced if it were not for the supervision of Dr. Hermann Brunner who provided a great quantity of immeasurable support, guidance, and motivation. His [seemingly unlimited] patience towards the author's mistakes is and was well-appreciated. In addition, casual conversations about professionalism, philosophy, idealism, and the personal value of hands-on skilled labour were a welcome relief to the rigors of this paper.

The author's experience with numerical analysis and programming is primarily a result of classes and consultations with Dr. Brunner and Dr. Phil Bording. Dr. Chun-Hua Ou is also deserving of mention.

The author would also like to express gratitude towards Dr. Bording (Husky Energy and IBM and/or ACOA-AIF), Dr. Brunner, the Mathematics and Statistics Department, and his parents as sources of support of the monetary type. This work would not have come to fruition without these.

Acknowledgements are also extended towards Gail Kenny, Michelle Miskell, the Department of Computational Science, and the Queen Elizabeth II Library. Without the assistance and guidance provided, the author would have had a much more difficult academic experience.

Appreciation for aid to the human side of existence during the work on this thesis — that is, friendship, comraderie, and understanding — is extended (but not limited) to Jeremy Rose, Stephanie Flynn, Dr. Scott Kenney, Frank Penney, and John E.E. Warren.

A general word of appreciation is extended to those persons and entities who expressed sympathy to the author's *student syndrome* — "I'm a student, so the cost of this item ..." — and adjusted their prices accordingly.

A final word appreciative note is directed to those who were omnipresent during these years: Nothandep Maes, Emil von Erswin, coffee, and myself[1].

[1] Yes, the author does, from time to time, display some degree of vanity.

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1 Introduction

This section will discuss some of the history surrounding the KdV equation, the physical context of it, and a synopsis of the rest of this paper.

1.1 From Discovery to First Numerical Methods

The development of the equation which is now termed the *KdV equation* can be shown to span a time period of approximately 60 years from time of observation until its present form. The most interesting solutions of the KdV equation consist of *solitons*, waves which maintain their profile as they travel and which retain their identities when interacting with other like waves.

The first recorded origin of the phenomena described can be traced back to 1834. A Scottish naval engineer (and lecturer at Edinburgh University) named John Scott Russell was at Union Canal at Hermiston, testing the efficiency of the propelling force of a canal boat with its resultant speed. Russell called the persisting wave a *Wave of Translation*[2], since the wave maintained its profile as it travelled down the canal. He presented the following from his paper [24] at the Fourteenth Meeting of the British Association for the Advancement of Science in 1844:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was

[2] *translation* quite literally means "carrying across"

my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

After experiencing the phenomena of the wave, Russell built in his back yard a thirty-foot basin in which he tried (and succeeded in) recreating such waves. He presented his results in the aforementioned Meeting, including sketches, which can be found in [24].

Then, a period of many years passed in which the mathematical explanation for this phenomena apparently went unexamined. In Russell's words, "[...] it now remained to the mathematician to predict the discovery after it had happened, i.e. to give an *a priori* demonstration *a posteriori.*" [24] In 1871, French mathematician Joseph Valentin Boussinesq published mathematical explanation [3] for the observations of Russell.[3] Five years later, Lord Rayleigh (John William Strutt) published his theoretical investigations of the *Wave of Translation.* Coincidentally, the two theories were very similar. From Rayleigh [23]: *"I have lately seen a memoir by M. Boussinesq [...] in which is contained a theory of the solitary wave very similar to that of this paper. So as far as our results are common, the credit of priority belongs of course to M. Boussinesq."*

Boussinesq essentially obtained the same equation that Korteweg and de Vries were to derive [16]. Two of his equations were

$$0 = u_t + \omega\frac{\partial u}{\partial x} + u\frac{\partial \omega}{\partial x}. \tag{1.1}$$

$$\omega(t,x) = \sqrt{gl} + \sqrt{gl}\left(\frac{3u}{4l} + \frac{l^2}{6u}\frac{\partial^2 u}{\partial x^2}\right) \tag{1.2}$$

where l is the undisturbed level of the liquid, u is the amplitude of the wave (vanishing at $x \to \infty$), and g is the constant of gravity [3].

Sixty-one years after the first observations of Russell, two men named Diederik Johannes Korteweg and Gustav de Vries (a previous doctoral student of the former) presented a paper which was based on the doctoral thesis of de Vries. It was because of the widespread fame of this paper that their equation was eventually to be called the "Korteweg-de Vries [KdV] equation".

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right) \tag{1.3}$$

[3]Boussinesq's paper is appended at the end of this paper.

where η is the elevation above the water level l, g is gravity, α is a small constant which allows for a correction to the wave velocity, and $\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$ (where ρ is density and T is the capillary tension at the surface of the fluid) [17].

The KdV equation can be derived using the work of Boussinesq [3]. Substituting Boussinesq's latter equation into the former, we remove the dependence on the wave speed $\omega(x,t)$ and retrieve

$$\frac{\partial u}{\partial t} + \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{2}{3}lu + \frac{l^3}{9}\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{1.4}$$

Boussinesq's results differ in the respect that Boussinesq's work involves a fixed reference frame (t,x). Using the transformation

$$\begin{aligned} \xi &= x - \left(\sqrt{gl} - \sqrt{\frac{g}{l}}\alpha\right)t \\ \tau &= t \end{aligned}$$

and, ignoring the added velocity [16], and setting the capillary tension T to zero, we get the KdV equation:

$$\frac{\partial u}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}u^2 + \frac{2}{3}\alpha u + \frac{1}{3}\sigma\frac{\partial^2 u}{\partial \xi^2}\right) \tag{1.5}$$

In 1965, Norman Zabusky and Martin Kruskal [29] performed some numerical experiments with the KdV equation. Specifically, they showed that certain initial conditions evolved into a finite set of travelling waves, each like the wave that Russell had observed. Most importantly, their results showed that a collision of two such travelling waves resulted in no change to the waves (other than phase shift). These particle-like behaviour of the waves is what influenced the two men to call the travelling waves "solitons" [7, p. 14-15]. This result is arguably what made renowned the work of Korteweg and de Vries. Kruskal states that, "Together with professor Norman J. Zabusky, I am the person who, more than anyone else, resuscitated (perhaps revitalized is the right word) the Korteweg-de Vries equation after its long period of, if not oblivion, at least neglect." [18, p. 1]

After the work of Gardner, Greene, Kruskal, and Miura, which involved scattering theory, exact solutions of the KdV equation could be found. The year 1975 brought about a paper by Bona and Smith [2] gave analytical solutions for the initial value problem for the KdV [2]. Fornberg and Whitham

(1978) devised a pseudospectral method which was the fastest method, to date, to the KdV equation [9]. It used three fast Fourier transforms per time step.

The 1980s contained the perhaps the most significant development with the numerical solution of the KdV equation. Sanz-Serna and Christie divised a modified Petrov-Galerkin finite element method, and compared their method to the leading methods that had been developed to date: Zabusky-Kruskal's finite difference method, the hopscotch method by Greig and Morris, the Petrov-Galerkin method, and their 'modified' Petrov-Galerkin method. Their 'modified' method was fourth order accurate in space. The errors of the methods are ordered from most to least.

Also in the 1980's, Taha and Ablowitz compared eight numerical methods [27]. They compared previous finite difference methods, finite element methods, Fourier transform methods, and pseudospectral methods against their 'local scheme'. While their results were significant, they apparently were unaware of the work that was accompished two years prior by Sanz-Serna and Christie, and thusly did not compare the aforementioned's modified Petrov-Galerkin method.

The last major numerical method comparison was presented in a paper by Nouri and Sloan in 1989 [22]. They compared six Fourier pseudospectral methods, including the 'local scheme' introduced by Taha and Ablowitz. The results of Nouri and Sloan showed that the 'local scheme' was more efficient for the one-soliton solution, but inferior to some other methods when applied to the two-soliton solution. They did not conduct experiments with three or more solitons.

In 1997, Brunner and Roth [4] used collocation in both spatial and temporal coordinates with B-splines. They compared the method to Fornberg and Whitham's pseudospectral method, finding that although collocation with two Gauss points was slower than the aforementioned pseudospectral method in general, it had a smaller error.

The most recent results are by far the most exciting. Yan and Shu in 2001 [28] used a local discontinuous Galerkin method. It was shown to obey conservation laws and was of order $k+1$ where k is the power of the polynomials used in the method. The authors even conveniently included a generalization to multiple spatial dimensions!

The items mentioned above are not comprehensive; rather, they respresent what the author of this paper considers significant enough to give a piecewise continuous approximation of the development of numerical solu-

tions to the KdV equation. The reader, if desiring to acquire a smoother curve with respect to its history, is directed towards the work of Kruskal[4], Tappert, Goda, Chen & Kerkhoven, and others.[5]

The author hopes that the mathematical contents of this paper are at least as interesting as the history lesson just told.

1.2 Physical Context

The equation in the 1895 paper by Korteweg and de Vries [17] looked different than the "modern" version that we write. Their model equation was

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\beta\frac{\partial^2\eta}{\partial x^2}\right)$$

where η is the elevation above the water level l, g is gravity, α is a small constant which allows for a correction to the wave velocity, and $\beta = \frac{l^3}{3} - \frac{Tl}{\rho g}$ (where ρ is density and T is the capillary tension at the surface of the fluid).

After years of development, the KdV equation is written in a more convenient form[6], with the physical constants eliminated:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.6}$$

where we are letting $u := u(t, x)$ denote the surface of a wave in shallow water. As usual, x will represent the physical position and, as expected, t will represent time. The subscripts refer to partial derivatives $u_t := \frac{\partial u}{\partial t}$, $u_x := \frac{\partial u}{\partial x}$, $u_{xxx} := \frac{\partial^3 u}{\partial x^3}$.

The KdV equation describes the movement of shallow water waves. This equation is the simplest equation which is both nonlinear and is dispersive [7, p. 6]. This equation has infinitely many conserved quantities (this remark will be revisited in §2.6). Single (solitary wave) soliton solutions of this

[4]His "suggested" scheme in the work of Taha & Ablowitz (1983)

[5]The work of these 'other' persons can be accessed by reading the second-generation references of this paper.

[6]The curious appearance of the 6 is present for ease of computation for the (complete) integrability of this equation.

dispersive partial differential equation are given by

$$u(t,x) = \frac{c}{2\cosh^2(k)} = \frac{c}{2}\operatorname{sech}^2(k) \tag{1.7}$$

where $k := \left(\frac{\sqrt{c}}{2}(x-ct)\right)$. The speed of the wave is given by c. In the case when $c = 0$ we have that $u(t,x) = 0$ for all t and all x. That is, as one would intuitively want and would physically demand, when the wave has no speed, the solution is zero everywhere. Alternatively, "a wave that isn't moving isn't a wave at all." A brief mental experiment with the function $u(t) = \operatorname{sech}^2(t)$ shows that if we let $t \to \pm\infty$, then the solution u approaches zero. Thus, solutions of the KdV equation asymptotically approach zero.

Many equations are both integrable and have soliton solutions[7], but the KdV equation is touted as the prototypical one [18, p. 5]. As an aside, note that a nonlinear equation can have a solitary wave solution that is not a soliton; for example, Burger's equation and the "traffic light problem". The solitary wave solution of the KdV equation, however, is a soliton.

These solutions are waves, and can interact strongly as if there had been no interaction. Zabusky and Kruskal denoted the word "soliton" to describe such waves. They chose such a term to emphasize the character of the waves, which behaved much like a particle such as a photon, which also retains its identity after a collision with another photon [7, p. 14-15]. Using the words of Kruskal [18, p. 5], "A soliton is a solitary traveling wave, but only in a solution of an equation special in the right way to support waves with the strong stability, or persistence of form through nonlinear interactions, that led Norman and me to give them such a particle-like name."

Waves described by the KdV equation have the following principles: First, solitary waves have the shape $\operatorname{sech}^2(x-ct)$. This shape has the property that as the argument $(x-ct)$ approaches infinity, the shape disappears. Secondly, a wave with a large amplitude travels faster than a wave with less amplitude. When discussing such waves, the terms 'larger' and 'smaller' are used in the comparative sense. There is no minimum of maximum for which this comparison does not apply. Thirdly, when a larger wave and smaller wave cross paths, the only change that occurs is that of displacement; their shapes and wave speeds do not change. Fourthly, if there is a sufficiently large initial mass of water, then there will be produced two or more independent solitary waves.

[7] such as the *sine-Gordon equation* $u_{tt} - u_{xx} + \sin u = 0$.

The term *soliton* is now defined. We refer to Drazin and Johnson [7, p. 15], who associate the word soliton with any solution of a nonlinear system which

1. Represents a wave of permanent form.
2. Is localized such that it approaches a constant as $x \to \infty$ for a fixed t.
3. Can interact with other such waves and still retain its identity.

Interacting solitons undergo displacement. The nonlinearity of the KdV equation allows for this interesting phenomena. Relative to the position in which the two waves would be with linear interaction, the taller wave has moved forward, and the shorter one has moved backward. The displacement of two interacting solitons is discussed in §2.7.

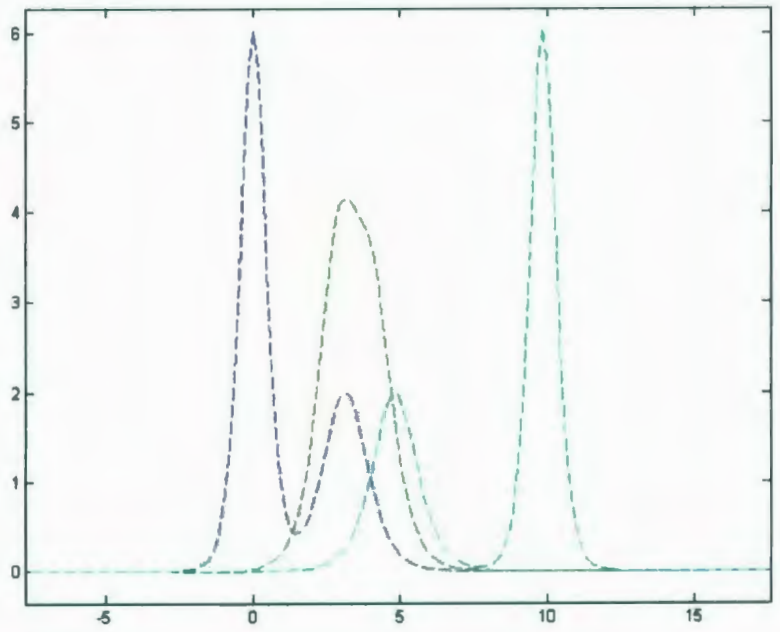

Figure 1: Illustration of two interacting solitons.

2 Soliton Solutions

There are several types of solutions to the KdV equation. Some of them are rational solutions, similarity solutions [7, p. 30-32], and the ones covered by the scope of this paper, the soliton solutions.

2.1 Soliton Solutions

Recall the common version of the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{2.1}$$

$$x \in (-\infty, \infty), t \geq 0 \tag{2.2}$$

where $u := u(t, x)$ denotes the surface of a wave in shallow water.

We seek travelling wave solutions of this equation. That is, we are looking for solutions $f(x - ct)$ and $g(x + ct)$ for f and g which correspond to d'Alembert's solution.[8] We choose to look at the right-travelling waves, $u(x, t) = f(\xi)$ where $\xi = x - ct$ where c is a constant which pertains to the speed of the wave. We have

$$-cf' - 6ff' + f''' = 0 \tag{2.3}$$

$$-cf - 3f^2 + f'' = A \tag{2.4}$$

for some arbitrary constant A. Using f' as an integrating factor, we retrieve

$$\frac{1}{2}(f')^2 = f^3 + \frac{1}{2}cf^2 + Af + B \tag{2.5}$$

for some other arbitrary constant B. If one assumes the boundary conditions that as $\xi \to \pm\infty$ the function f and all of its derivatives $\to 0$, then this implies that our arbitrary constants are both zero; thus

$$(f')^2 = f^2(2f + c) \tag{2.6}$$

[8] The reader may recall that d'Alembert's solution to the wave equation $u_{tt} = c^2 u_{xx}$ is given by any sum of the form $u(x, t) = f(x - ct) + g(x + ct)$.

Thus a real solution exists if $(f')^2 \geq 0$; that is, $2f + c \geq 0$. Continuing with the solution, we have

$$\begin{aligned}
(f')^2 &= f^2(2f+c), \\
|f'| &= |f|(2f+c)^{1/2}, \\
f' &= \pm f(2f+c)^{1/2}, \\
\frac{df}{d\xi} &= \pm f(2f+c)^{1/2}, \\
\frac{df}{f(2f+c)^{1/2}} &= \pm\, d\xi, \\
\int \frac{df}{f(2f+c)^{1/2}} &= \pm \int d\xi. \qquad (2.7)
\end{aligned}$$

Using the substitution $f = -\frac{c}{2}\ \mathrm{sech}^2(\theta)\ \ (c \geq 0)$, we get that

$$\begin{aligned}
\int \frac{df}{f(2f+c)^{1/2}} &= \int \frac{c\ \mathrm{sech}^2(\theta)\tanh(\theta)\ d\theta}{-\frac{1}{2}c^{\frac{3}{2}}\ \mathrm{sech}^2(\theta)\left(-\ \mathrm{sech}^2(\theta)+1\right)^{1/2}}, \\
&= -\frac{2}{\sqrt{c}}\int d\theta, \\
\Rightarrow \theta &= \pm\frac{\sqrt{c}}{2}\int d\xi, \\
\theta &= \pm\frac{\sqrt{c}}{2}\xi, \\
\mathrm{sech}^2(\theta) &= \mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\xi\right), \\
-\frac{c}{2}\ \mathrm{sech}^2(\theta) &= -\frac{c}{2}\ \mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\xi\right), \\
f(\xi) = f(x-ct) &= -\frac{c}{2}\ \mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct-x_0)\right), \qquad (2.8)
\end{aligned}$$

where the constant x_0 is the phase shift which denotes the position of the peak at $f(x,0)$. Using typical notation of a solution of the KdV equation, define $u(x,t) := f(x,t)$. Thus, we have found a wave $u \in C^\infty$ of permanent

form which vanishes[9] at $x \to \infty$:

$$\begin{aligned} u_t - 6uu_x + u_{xxx} &= 0, \quad x \in (-\infty, \infty),\, t > 0 \\ u(x,0) &= -\frac{c}{2}\ \mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - x_0)\right) \\ u(x,t) &= 0 \ \text{ as } x \to \pm\infty. \end{aligned} \tag{2.9}$$

2.2 Uniqueness

As is typical of a uniqueness proof, first assume that there are two solutions, u and v, both which satisfy the KdV and have the same initial condition $u(x,0) = v(x,0) = f(x)$. Subtracting one from the other, we have

$$\begin{aligned} \frac{\partial}{\partial t}(u - v) &= u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x} + \frac{\partial^3(u-v)}{\partial x^3} \\ &= u\frac{\partial (u-v)}{\partial x} + (u-v)\frac{\partial v}{\partial x} + \frac{\partial^3(u-v)}{\partial x^3} \\ \iff \frac{\partial w}{\partial t} &= u\frac{\partial w}{\partial x} + w\frac{\partial v}{\partial x} + \frac{\partial^3 w}{\partial x^3} \end{aligned}$$

where $w(x,t) := u(x,t) - v(x,t)$. Multiply (2.10) by w and integrate with respect to x:

$$\begin{aligned} w\frac{\partial w}{\partial t} &= wu\frac{\partial w}{\partial x} + w^2\frac{\partial v}{\partial x} + w\frac{\partial^3 w}{\partial x^3}, \\ \int_{-\infty}^{\infty} ww_t\,dx &= \int_{-\infty}^{\infty} ww_x u + w^2 v_x + ww_{xxx}\,dx, \\ \int_{-\infty}^{\infty} (u-v)(u-v)_t\,dx &= \int_{-\infty}^{\infty} w^2 v_x + ww_x u + ww_{xxx}\,dx, \end{aligned} \tag{2.10}$$

[9] Technically, the wave is asymptotically constant at infinity; we just chose that constant to be zero.

$$\begin{aligned}
\int_{-\infty}^{\infty} u_t u + v_t v - (u_t v + w v_t)\, dx &= \int_{-\infty}^{\infty} w^2 v_x\, dx, \\
&+ \left(\frac{1}{2} w^2 u|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2} w^2 u_x\, dx \right), \\
&+ \left(w w_{xx}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w_{xx} w_x\, dx \right), \\
\int_{-\infty}^{\infty} \frac{d}{dt} \left(\frac{u^2}{2} + \frac{v^2}{2} - uv \right) dx &= \int_{-\infty}^{\infty} w^2 v_x - \frac{1}{2} w^2 u_x\, dx - \int_{-\infty}^{\infty} w_{xx} w_x\, dx, \\
\frac{d}{dt} \int_{-\infty}^{\infty} \frac{1}{2} (u - v)^2\, dx &= \int_{-\infty}^{\infty} w^2 v_x - \frac{1}{2} w^2 u_x\, dx - \frac{1}{2} (w_x)^2|_{-\infty}^{\infty}, \\
\frac{d}{dt} \int_{-\infty}^{\infty} \frac{1}{2} w^2\, dx &= \int_{-\infty}^{\infty} w^2 (v_x - \frac{1}{2} u_x)\, dx. \qquad (2.11)
\end{aligned}$$

where we used the fact that $w \to 0$ as $x \to \infty$ since u and v each approach zero as $x \to \infty$, which are our boundary conditions.

Defining $E(t) = \int_{-\infty}^{\infty} \frac{1}{2} w^2\, dx$, we have

$$\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{\infty} \frac{1}{2} w^2\, dx &= \int_{-\infty}^{\infty} w^2 (v_x - \frac{1}{2} u_x)\, dx, \\
\frac{d}{dt} E(t) &= \int_{-\infty}^{\infty} w^2 (v_x - \frac{1}{2} u_x)\, dx, \\
&= \int_{-\infty}^{\infty} \frac{1}{2} w^2 \cdot 2 (v_x - \frac{1}{2} u_x)\, dx, \\
&\leq \int_{-\infty}^{\infty} \frac{1}{2} w^2\, dx \cdot \int_{-\infty}^{\infty} 2 (v_x - \frac{1}{2} u_x)\, dx, \\
&= E(t) \cdot 2 \int_{-\infty}^{\infty} (v_x - \frac{1}{2} u_x)\, dx, \\
&= m E(t). \qquad (2.12)
\end{aligned}$$

where $m := 2 \max |v_x - \frac{1}{2} u_x|$. We clearly have an ordinary differential equation,

$$\frac{dE(t)}{dt} = m E(t), \qquad (2.13)$$

which of course has the solution $E(t) = E(0) e^{mt}$. If our initial condition vanishes, then $E(t)$ is also forced to zero. Since we had assumed that

$u(x,0) = v(x,0) = f(x)$, we have that $w(x,0) = u(x,0) - v(x,0) = 0$ and thus that

$$E(0) = \frac{1}{2}\int_{-\infty}^{\infty} w^2(x,0)\,dx = 0 \tag{2.14}$$

From above, $E(0) \to 0 \Rightarrow E(t) \to 0$. Thus, $w(x,t) = 0$. Explicitly, we have $w(x,t) = u(x,t) - v(x,t) = 0$. Therefore $u(x,t) = v(x,t)$, meaning that the two solutions are identical, and that a solution of the KdV equation is unique.

2.3 Construction and Behaviour of Soliton Solutions

A solution which has the form of multiple solitons is actually set from the initial condition. If the initial profile can be written in the form

$$u(x,0) = -n(n+1)\,\mathrm{sech}^2(x) \tag{2.15}$$

for n a positive integer, then the solution will evolve into a system of exactly n solitons [7][also see §2.7].

In the case that the coefficient for the initial condition cannot be constructed in this manner, then a dispersive wave will occur [7, p. 83, 45-48]. We leave the soliton solutions with a remark taken from Drazin and Johnson [7]. There is a dispersive-wave-only solution which occurs only when the initial profile is a positive sech^2 function, ie

$$u(x,0) = a\ \ \mathrm{sech}^2(x) \tag{2.16}$$

for some $a \in \mathbb{R}$.

2.4 Rational Solutions

Another class of solutions of the KdV equation are called *rational solutions*, and all of them are singular. Starting again from the KdV equation

$u_t - 6uu_x + u_{xxx} = 0$, assume that the solution u is a function of only x, i.e. $u = u(x)$, and that u and its derivatives $\to 0$ as $x \to \pm\infty$. Then we have that

$$-6uu' + u''' = 0 \tag{2.17}$$

We integrate this and then solve:

$$\begin{aligned}
u''' &= 6uu' \\
u'' &= 3u^2 \\
2u'' &= 6u^2 \\
2u'' \cdot u' &= 6u^2 \cdot u' \\
(u')^2 &= 2u^3 \\
u' &= \sqrt{2}u^{\frac{3}{2}} \\
\int \frac{u^{-\frac{3}{2}}}{\sqrt{2}}\, du &= \int dx \\
-\sqrt{2}u^{-\frac{1}{2}} &= x \\
u(x) &= u(x,t) = \frac{2}{x^2}
\end{aligned} \tag{2.18}$$

which is clearly singular at $x = 0$.

The "next" rational solution, which corresponds to the solution which involves two solitons [7], is

$$u(x,t) = 6x(x^3 - 24t)/(x^3 + 12t)^2 \tag{2.19}$$

which becomes singular when $x^3 = -12t$.

As one might infer at this point, there is a rational solution for each solution which involves any number of solitons. The rational solutions can be found by taking an appropriate limit in each initial profile which corresponds to n solitons [7], where each initial profile is as mentioned above in §2.3.

2.5 Conserved Quantities

Starting again from the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0$$

we note that it is already in *conservation form*

$$\frac{\partial}{\partial t}(T) + \frac{\partial}{\partial x}(X),$$

where $T := u$ and $X := u_{xx} - 3u^2$.

If T and X_x are integrable and u satisfies the KdV equation, then

$$\int_{-\infty}^{\infty} u \, dx = A$$

for some constant A. This condition applies for all solutions of the KdV equation which vanish at infinity.

Multiply the KdV equation by u and retrieve

$$uu_t - 6u^2u_x + uu_{xxx} = 0, \tag{2.20}$$

which is equivalent to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2 - 2u^3\right) = 0,$$

which is already in conservation form $\frac{\partial}{\partial t}(T) + \frac{\partial}{\partial x}(X)$ where T and X are now defined by $T := \frac{1}{2}u^2$ and $X := uu_{xx} - \frac{1}{2}u_x^2 - 2u^3$. Thus

$$\int_{-\infty}^{\infty} u^2 \, dx = A \tag{2.21}$$

for some constant A.

2.6 An Infinity of Conservation Laws

Surprisingly, there are an unlimited number of conservation laws for the KdV. Perhaps even more surprising is the situation surrounding the proof

of the claim. The discovery was made rather simultaneously in a rigorous sense of the term; Kruskal and Miura were in one location, having proven the existence of such laws. As they were examining the result, Gardner called to inform the two gentlemen of his (different) proof of the existence of the laws.

Starting from the Gardner transformation [7, p. 92-95],

$$u = w + \varepsilon w_x + \varepsilon^2 w^2, \tag{2.22}$$

we apply this to the KdV equation:

$$\begin{aligned} u_t - 6uu_x + u_{xxx} = w_t &+ \varepsilon w_{xt} + 2\varepsilon^2 w w_t \\ &- 6(w + \varepsilon w_x + \varepsilon^2 w^2)(w_x + \varepsilon w_{xx} + 2\varepsilon^2 w w_x) \\ &+ w_{xxx} + \varepsilon w_{xxxx} + 2\varepsilon^2 (w w_x)_{xx}. \end{aligned}$$

The u in the above equation is a solition of the KdV equation if w is a solution of the equation

$$w_t - 6(w + \varepsilon^2 w^2) w_x + w_{xxx} = 0,$$

which is already in conservation form (again)

$$\frac{\partial}{\partial t}(w) + \frac{\partial}{\partial x}(w_{xx} - 3w^2 - 2\varepsilon^2 w^3)$$

(the *Gardner equation*), that is, $\frac{\partial}{\partial t}(T) + \frac{\partial}{\partial x}(X)$ with $T := w$ and $X := w_x x - 3w^2 - 2\varepsilon^2 w^3$. Noting that $w \to u$ as $\varepsilon \to 0$, we represent w as an asymptotic expansion. As $\varepsilon \to 0$, we have a formal power series in ε

$$\begin{aligned} w(x,t;\varepsilon) &\sim \sum_{n=0}^{\infty} \varepsilon^n w_n(x,t) \\ &= w_0 + \varepsilon w_1 + \varepsilon^2 w^2 + \ldots \\ &= u - \varepsilon u_x - \varepsilon^2 (u^2 - u_{xx}) + \ldots \end{aligned}$$

with the coefficient of each power of ε being a conservation law for the KdV. Also, only the coefficients of the even powers of ε are nontrivial conservation laws. For a full proof of these points, see the work of Gardner et al. [10].

2.7 Interacting Solitons

Using inverse scattering [7, p.72-76], the two-soliton solution can be written as

$$u(x,t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{(3\cosh(x - 28t) + \cosh(3x - 36t))^2}. \tag{2.23}$$

The interaction of these two solitons can be seen by introducing $\xi = x - 16t$. We substitute this in (2.23) and get

$$u(x,t) = -12\frac{3 + 4\cosh(2\xi + 24t) + \cosh(4\xi)}{(3\cosh(\xi - 12t) + \cosh(3\xi + 12t))^2},$$

which can be expanded for $t \to \pm\infty$. If a wave moving at speed 16 exists, then we have

$$u(x,t) \sim -8\,\mathrm{sech}^2(2\xi \mp \frac{1}{2}log3) \quad \text{as } t \to \pm\infty,\ \xi = x - 16t,$$

and for a wave which exists and moves at speed 4, we have

$$u(x,t) \sim -2\,\mathrm{sech}^2(\eta \pm \frac{1}{2}log3) \quad \text{as } t \to \pm\infty,\ \eta = x - 4t.$$

Since the error terms are exponentially small [7], these two asymptotic waves can be combined:

$$u(x,t) \sim -8\,\mathrm{sech}^2(2\xi \mp \frac{1}{2}log3) - 2\,\mathrm{sech}^2(\eta \pm \frac{1}{2}log3), \tag{2.24}$$
$$\text{as } t \to \pm\infty,\ \xi = x - 16t,\ \eta = x - 4t.$$

Now we have a solution at infinity of two solitary waves with explicit phase shifts. The larger wave is shifted further ahead at the distance $x = \frac{1}{2}\log 3$ and the smaller wave is shifted backwards at the distance $x = \frac{1}{2}\log 3$, as shown by (2.24).

For the case of N solitons, the solution of the Marchenko equation [7, p. 56-60] yields the algebraic system

$$AL + B = 0,$$

where B_n and L_n are column vectors each of length N, where $B_n = c_n^2(0)\exp(8n^3t - nx)$ and each c_i is the coefficient of discrete eigenvalues $\Psi(x)_i \sim c_i \exp(-\kappa_i x)$ (as $x \to 0$) where $\kappa_i \neq \kappa_j$ for any i, j. The matrix A is of size NxN with elements

$$A_{mn} = \delta_{mn} + c_m^2 \frac{\exp(-(\kappa_m + \kappa_n)x)}{\kappa_m + \kappa_n},$$

where δ_{mn} is the Kronecker delta function.

The solution of this system is [7]

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} log\,|A|\,.$$

The asymptotic form of the solution is

$$u(x,t) \sim -2n^2 \operatorname{sech}^2(n(x - 4n^2t) \mp x_n),$$
$$\exp(2x_n) = \prod_{m=1, m\neq n}^{N} \left|\frac{n-m}{n+m}\right|^{sgn(n-m)},$$

for $n = 1 : N$. Again, the error terms are exponentially small and the N asymptotic waves can be written as

$$u(x,t) \sim -2\sum_{n=1}^{N} \operatorname{sech}^2((nx - 4n^2t) \mp x_n) \quad \text{as } t \to \pm\infty. \tag{2.25}$$

Thus, there are separate solitons which comprise the solution, with the largest wave at the front of the wavetrain, followed by the next largest, et cetera, such that the waves are ordered from largest to smallest due to their velocities [7]. The phase shifts in each wave can be computed in a similar manner as shown in the beginning of this section §2.7.

3 Numerical Schemes for KdV

It is important to note that some authors write the KdV equation with a change of coefficient in the nonlinear term. For instance, $u_t + 6uu_x + u_{xxx}$ is used instead of $u_t - 6uu_x + u_{xxx}$. In the case that the literature makes a change in this manner, then the references to the KdV equation with the nonlinear term written as either $-6uu_x$, $+6uu_x$, $-uu_x$, or $+uu_x$ will be called KdV, +KdV, KdV_-, and KdV_+ respectively. In the case that the KdV_+ or the +KdV equation is used, then the initial conditions will have their signs changed with respect to how they are written in the rest of this paper.

3.1 Explicit Finite Difference Methods

In both the implicit and explicit finite difference methods, we use a mesh on the spatial domain of N equidistant points, with the distance between each point being $h := \Delta x$ (unless otherwise stated). The temporal domain is discretized uniformly in steps of $k := \Delta t$. The subscripts of a function u refer to the spatial steps and the superscripts refer to the temporal steps. For example, $u_{j+2}^{n-1} \approx u(x + 2\Delta x, t - \Delta t) = u(x + 2h, t - k)$.

3.1.1 Zabusky & Kruskal

Zabusky and Kruskal were the first to publish numerical results on the interaction of solitons on the KdV_+ equation.

$$\begin{aligned} u_t + uu_x + \epsilon^2 u_{xxx} &= 0, \\ x \in (-\infty, \infty), t &> 0, \end{aligned} \tag{3.1}$$

with initial condition of $u(x, 0) = \cos(\pi x)$.

The presence of the ϵ^2 often appears in literature concerning the numerical solution of the KdV problem. The main focus of their paper was the interaction of the solitons and the recurrence of the initial condition that

they had set: "In conclusion, we should emphasize that [...] all [of] the solitons arrive *almost* in the same phase and almost reconstruct the initial state through the nonlinear interaction. This process proceeds onwards, and [...] on again has a "near recurrence" which is not as good as the first recurrence." [29, p. 242] This "recurrence" was revisited by Goda in 1977 [13].

As we will not delve into the derivation of the method, the reader is assumed to know *central finite difference* approximations to at least the third derivative:

$$\begin{aligned} f'(x_0) &\approx \frac{1}{2h}(f_1 - f_{-1}) \\ f'''(x_0) &\approx \frac{1}{2h^3}(f_2 - 2f_1 + 2f_{-1} - f_{-2}) \end{aligned}$$

for a function f at a point x_0 with uniform mesh spacing $h := \Delta x$. The subscripts y refer to the values $f(x_0 + yh)$.

We seek a finite difference approximation of second order in both variables. Zabusky and Kruskal chose to use a three-point average for the approximation of u, that is

$$f(x_0) \approx \frac{1}{3}(f_1 + f_0 + f_{-1}).$$

Using approximations as described, we have the following finite difference method for (3.1):

$$\begin{aligned} \frac{1}{2\Delta t}(u_j^{n+1} - u_j^{n-1}) &= -u \cdot \frac{1}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) \\ &\quad - \epsilon^2 \frac{1}{2\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n), \\ \frac{1}{2\Delta t}(u_j^{n+1} - u_j^{n-1}) &= -\left(\frac{u_{j+1}^n + u_j^n + u_{j-1}^n}{3}\right) \cdot \frac{1}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) \\ &\quad - \epsilon^2 \frac{1}{2\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n), \\ u_j^{n+1} - u_j^{n-1} &= -\frac{1}{3}(u_{j+1}^n + u_j^n + u_{j-1}^n) \cdot \frac{\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n) \\ &\quad - \epsilon^2 \frac{\Delta t}{\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n), \\ u_j^{n+1} &= u_j^{n-1} - \frac{\Delta t}{3\Delta x}(u_{j+1}^n + u_j^n + u_{j-1}^n)(u_{j+1}^n - u_{j-1}^n) \\ &\quad - \epsilon^2 \frac{\Delta t}{\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n). \end{aligned} \tag{3.2}$$

Thus we have an explicit finite difference method for the KdV equation.[10] The method has a truncation error of $O((\Delta t)^2)+O((\Delta x)^2)$ and requires that

$$\frac{\Delta t}{\Delta x}\left|-2u_{max}+\frac{1}{(\Delta x)^2}\right| \leq \frac{2}{3\sqrt{3}}$$

in order to be stable [27, p.233].

As one may infer from the stability condition, this method unfortunately needs a small time step, and is thus one of the slowest respectable methods available to date. To its credit, it is an accurate method, and due to this fact it is a benchmark against which other methods are compared.

3.2 Implicit Finite Difference Methods

3.2.1 Greig & Morris's "Hopscotch" method

Again looking at the KdV equation written as $u_t + uu_x + u_{xxx}$, observe that

$$uu_x = \frac{1}{2}(u^2)_x. \tag{3.3}$$

Now we approximate $\frac{1}{2}u_x^2$ using a central difference approximation. Define $w(x,t) := \frac{1}{2}u^2(x,t)$. Then we have

$$\begin{aligned}\left(\frac{u^2}{2}\right)_x &=: w_x \\ &\approx \frac{1}{2\Delta x}(w_{j+1}^n - w_{j-1}^n) \\ &= \frac{1}{2\Delta x}\left(\frac{(u_{j+1}^n)^2}{2} - \frac{(u_{j-1}^n)^2}{2}\right),\end{aligned}$$

which has a truncation error of $O(\Delta x^2)$ [14].

[10] For the initial time step, one can use the uncentered scheme $u_j^1 = u_j^0 - \frac{\Delta t}{6\Delta x}(u_{j+1}^0 + u_j^0 + u_{j-1}^0)(u_{j+1}^0 - u_{j-1}^0) - \epsilon^2\frac{\Delta t}{2(\Delta x)^3}(u_{j+2}^0 - 2u_{j+1}^0 + 2u_{j-1}^0 - u_{j-2}^0)$.

Using a forward difference scheme in time, and a central difference with the linear term, we have the explicit method

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(w_{j+1}^n + w_{j-1}^n\right)$$
$$-\frac{\Delta t}{2(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right),$$

where $w := \frac{u^2}{2}$.

Eventually (see [14]), we find the algebraic system

$$\mathbf{A}\mathbf{v}^{m+1} = \mathbf{K},$$

where $\mathbf{A} = \begin{bmatrix} 1 & \frac{p\epsilon}{2h^2} & 0 & \cdots & 0 \\ -\frac{p\epsilon}{2h^2} & 1 & \frac{p\epsilon}{2h^2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\frac{p\epsilon}{2h^2} & 1 & \frac{p\epsilon}{2h^2} \\ 0 & \cdots & 0 & -\frac{p\epsilon}{2h^2} & 1 \end{bmatrix}$, $\mathbf{v}^{m+1} = \begin{bmatrix} v_1 \\ v_3 \\ \vdots \\ v_{N-2} \end{bmatrix}^{m+1}$,

$\mathbf{K} = [K_1, K_3, ..., K_{N-2}]$ and $p = \frac{\Delta t}{\Delta x} = \frac{k}{h}$.

This method has a truncation error of $O((\Delta t)^2) + O((\Delta x)^2)$, which is the same truncation error as the Zabusky-Kruskal finite difference method. The method has a stability requirement of

$$\frac{\Delta t}{(\Delta x)^3} \leq \left|\frac{1}{(\Delta x)^2 u_{max} - 2}\right|$$

which is less restrictive than Zabusky-Kruskal's method.

3.2.2 Goda's Scheme

Goda's scheme [13] uses a forward difference scheme in time, a central difference scheme in the u_{xxx} term, and a combination of methods for the

nonlinear term uu_x.

$$\begin{aligned} u_x u &\approx \left(\frac{1}{2\Delta x}(u_{j+1} - u_{j-1})\right) u_j \\ &= \frac{1}{2\Delta x}(u_{j+1}u_j - u_{j-1}u_j) \\ &= \frac{1}{2\Delta x}(u_{j+1}u_{(j1)} - u_{j-1}u_{(j2)}), \end{aligned}$$

where the references to time steps have been temporarily omitted for the sake of simplicity. The two occurances of the function u are approximated differently. The first is approximated by a forward explicit average $u_{(j1)} \approx \frac{1}{2}(u_j + u_{j+1})$ and the second by a backward explicit average $u_{(j2)} \approx \frac{1}{2}(u_j + u_{j-1})$. Goda's method is largely implicit, with the exception of the approximations for $u_{(1)}$ and $u_{(2)}$. Goda's approximation for the KdV equation $(u_t + uu_x + u_{xxx} = 0)$ is

$$\begin{aligned} \frac{1}{\Delta t}(u_j^{n+1} - u_j^n) + \frac{1}{\Delta x}\left(u_{j+1}^{n+1}(u_j^n + u_{j+1}^n) - u_{j-1}^{n+1}(u_j^n + u_{j-1}^n)\right) \\ + \frac{1}{2(\Delta x)^3}\left(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{n-1}^{n+1} - u_{j-2}^{n+1}\right) = 0. \end{aligned} \tag{3.4}$$

The method has a truncation error of $O(\Delta t) + O((\Delta x)^2)$ and is unconditionally stable, meaning that stability is achieved for any choice of Δt.

3.3 Fourier / Pseudospectral Methods

The Fourier/pseudospectral method is a global approximation method which has several advantages over finite difference methods. Due to the properties of the Fourier transform F for derivatives for a function f, that is $F\left(\frac{d^n f}{dx^n}\right) = (ik)^n F(f)$, no approximation for the spatial derivatives is necessary. Thus, a smaller number of grid points are required for the algorithm. Overall, the computing memory and number of computations can be reduced significantly in a given problem. Detailed descriptions can be found in the books [28], [15], and especially [8].

3.3.1 Fornberg & Whitham's method

This method uses the Fourier transform. Because of this, the spatial domain $[-p, p]$ is normalized to $[0, 2\pi]$ under the change of variable $x \to x\pi/p + \pi$. The normalized KdV equation is

$$u_t + 6\frac{\pi}{p}uu_x + \frac{\pi^3}{p^3}u_{xxx} = 0, \qquad u(0,t) = u(2\pi,t) \tag{3.5}$$

(see [9]). For N equally spaced points, define $\Delta x = 2\pi/N$. On the discrete Fourier space, the solution $u(x,t)$ takes the form

$$\begin{aligned} Fu = \hat{u} &= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} u(j\Delta x, t)e^{-2\pi ijk/N}, \\ u = F^{-1}(F(u)) = F^{-1}\hat{u} &= \frac{1}{\sqrt{N}}\sum_{k} \hat{u}(k,t)e^{2\pi ijk/N}, \\ k &= -\frac{N}{2}, ..., -1, 0, 1, ..., \frac{N}{2} - 1. \end{aligned}$$

Using a leapfrog time step, we apply the Fourier transform to the spatial domain.

$$\begin{aligned} F^{-1}F(u_{xxx}) &= F^{-1}(ik)^3F(u) = -iF^{-1}(k^3F(u)) \\ F^{-1}F(u_x) &= iF^{-1}(kF(u)) \\ &\Rightarrow uu_x = iuF^{-1}(kF(u)), \end{aligned}$$

where we have used $F\left(\frac{d^n u}{dx^n}\right) = (ik)^n F(u)$.

We can now approximate the solution:

$$\begin{aligned} u_t &+ 6\frac{\pi}{p}uu_x + \frac{\pi^3}{p^3}u_{xxx} = 0 \qquad (3.6) \\ u_t &+ 6iu\frac{\pi}{p}F^{-1}(kF(u)) - i\frac{\pi^3}{p^3}F^{-1}(k^3F(u)) = 0 \\ \frac{1}{2\Delta t}(u_j^{n+1} - u_j^{n-1}) &+ 6iu\frac{\pi}{p}F^{-1}(kF(u)) - i\frac{\pi^3}{p^3}F^{-1}(k^3F(u)) = 0 \\ u_j^{n+1} - u_j^{n-1} &+ 2i\Delta t\{6u_j^n\frac{\pi}{p}F^{-1}(kF(u)) - \frac{\pi^3}{p^3}F^{-1}(k^3F(u))\} = 0. \end{aligned}$$

Thus we have obtained a pseudospectral method for the KdV equation. However, Fornberg and Whitham's original work included a modification to the final term:

$$-2i\Delta t\frac{\pi^3}{p^3}F^{-1}(k^3F(u)) \longrightarrow -2iF^{-1}\left\{sin\left(\frac{\pi^3k^3}{p^3}\Delta t\right)F(u)\right\}.$$

This modification is intended to yield higher accuracy at high wavenumbers, and requires three Fourier transforms per time step, just as before the modification.

The truncation error of this method is of order $(O(\Delta t)^2 + O(\Delta x))$. The requirement for the first pseudospectral method is $\Delta t < (\Delta x)^3/\pi^3$, and for the second is $\Delta t < 3(\Delta x)^3/2\pi^2$ [9, p.376]. Since $\Delta x < 1$, it is known that $(\Delta x)^3 < (\Delta x)^2$, that is, a larger time step Δt can be used, which saves on computation time.

3.3.2 Taha & Ablowitz

Taha and Ablowitz compared (1982) six different methods in the numerical computation of the KdV equation [27]:

- Zabusky-Kruskal scheme
- Greig-Morris Hopscotch method
- Scheme due to Goda
- Proposed local scheme
- Scheme suggested by Kruskal
- Split step Fourier method by Tappert
- Pseudospectral method by Fornberg and Whitham

The scheme suggested by Kruskal was based on his suggestion for the equation $u_t + u_{xxx} = 0$ in which the dispersion term u_{xxx} was to be approximated by

$$\frac{1}{2(\Delta x)^3}(u_{j+2}^{n+1} - 3u_{j+1}^{n+1} + 3u_j^{n+1} - u_{j-1}^{n+1})$$
$$+\frac{1}{2(\Delta x)^3}(u_{j+1}^{n} - 3u_j^{n} + 3u_{j-1}^{n} - u_{j-2}^{n}).$$

The *proposed local scheme* was based on an inverse scattering transform (see [7]). The authors' results compared the performance of their *proposed local scheme* to the other aforementioned methods on only the single soliton solution. Their conclusion, somehow not unpredictably, showed that the *proposed local scheme* was the most accurate. Seven years later, Nouri and Sloan [22] tested the *proposed local scheme* on the two soliton solution. It was dominated by a different Fourier pseudospectral method by Chan and Kerkhoven [5]. "The results show that the local scheme is more efficient than the pseudospectral scheme on the 1-soliton problem, but less efficient on the more difficult 2-soliton problem. However, the differences in computing times are not large, and the results support the claim by Taha & Ablowitz that finite difference schemes based on the inverse scattering transform (IST) provide good approximations for equations which are solvable by the IST."

3.3.3 Semi-Implicit Scheme (Chan & Kerkhoven)

One of the methods authored by Chan and Kerkhoven uses a Crank Nicholson method for the linear term u_{xxx} and a leapfrog method for the nonlinear term written as $-3(u^2)_x$. To date, the fastest method with which to solve the KdV is Chan & Kerkhoven's semi implicit scheme.

The linear term is approximated as

$$\begin{aligned} u_{xxx} &\approx \frac{1}{2}(u_j^{n+1} + u_j^n)_{xxx} \\ F(u_{xxx}) &\approx \frac{1}{2}F\{(u_j^{n+1} + u_j^n)_{xxx}\} \\ &= -ik^3\frac{1}{2}F(u_j^{n+1} + u_j^n), \end{aligned}$$

and the nonlinear term as

$$-3F((u^2)_x) \approx -3ik\frac{\pi}{p}F((u_j^n)^2),$$

which yields the following:

$$\begin{aligned}
\frac{1}{2\Delta t}F(u_j^{n+1} - u_j^{n-1}) - 3ikF((u_j^n)^2) - ik^3F(u_j^{n+1} + u_j^n) &= 0, \\
F(u_j^{n+1} - u_j^{n-1}) - 6ik\Delta tF((u_j^n)^2) - 2ik^3\Delta tF(u_j^{n+1} + u_j^n) &= 0, \\
F(u_j^{n+1}) - F(u_j^{n-1}) - 6ik\Delta tF((u_j^n)^2) - 2ik^3\Delta t(F(u_j^{n+1}) + F(u_j^n)) &= 0,
\end{aligned}$$

$$(1 - 2ik^3\Delta t)F(u_j^{n+1}) = F(u_j^{n-1}) + 6ik\Delta tF((u_j^n)^2) + 2ik^3\Delta tF(u_j^n),$$

$$\begin{aligned}
F(u_j^{n+1}) &= \kappa(k)\left(F(u_j^{n-1}) + 6ik\Delta tF((u_j^n)^2) + 2ik^3\Delta tF(u_j^n)\right), \\
u_j^{n+1} &= F^{-1}\left\{\kappa(k)\left(F(u_j^{n-1}) + 6ik\Delta tF((u_j^n)^2) + 2ik^3\Delta tF(u_j^n)\right)\right\},
\end{aligned}$$

where $\kappa(k) := \frac{1}{(1-2ik^3\Delta t)}$. This method requires two FFT per time step and requires that

$$(\Delta t)^2 < \frac{3\sqrt{3}}{2}\frac{1}{|\alpha|^3}$$

(where α is the coefficient on the nonlinear term) for stability.

3.4 Collocation

3.4.1 Collocation in Space and Time

Brunner and Roth [4] considered the use of collocation in both space and time to solve equations of the form

$$\begin{aligned}
u_t + A(u) &= 0, \quad t \in (0, T], \\
u(x, 0) &= \phi(x), \quad x \in \Omega := I\!R,
\end{aligned}$$

where $A(u)$ is a linear or nonlinear operator of u and its derivatives in the spatial domain. The spatial derivatives are assumed to approach zero as

$|x| \to \infty$. The solution $u(x,t)$ is approximated by a linear combination of time-dependent basis functions $U(x,t)$ on $[a,b] \times [0,T]$. The spatial domain Ω is partitioned as $\Pi^x := \{\Pi^x_M : a = x_0 < x_1 < ... < x_M = b\}$, with $h_m := x_{m+1} - x_m$ and $h := max\{h_m\}$, $m = 0, 1, ..., M-1$.

$$U(x,t) = \sum_{j=1}^{d} B_j(x) w_j(t),$$

where $d := M\delta$ and the B_j are a B-spline basis $\beta := \{B_j(x) : 1 \leq j \leq d\}$ with the first and last $(p+1-\delta)/2$ removed. The δ is chosen $\delta \in \{1, 2, ..., p+1\}$ where p is the maximum degree of the polynomials in the polynomial space π_p.

Approximating the system, we have

$$\begin{aligned} \mathbf{B}\mathbf{w}' + A(\mathbf{B}\mathbf{w}) &= 0, \\ \mathbf{B}\mathbf{w} &= \Phi \quad \text{initial condition,} \end{aligned}$$

where $\mathbf{B} \in I\!R^{d \times d}$ is the matrix with $B_j(\hat{x}_i)$ in row i and column j, and $\mathbf{w} := (w_1, ..., w_d)^T$.

On each subinterval $\Omega_m := [x_m, x_{m+1}]$, set $\hat{x}_{m\delta+i} := x_m + c_i h_m$, $i = 1, ..., \delta$, and collocation parameters $0 \leq c_1 < c_2 < ... < c_\delta \leq 1$. To avoid instabilites, the introduction of additional meshpoints are needed beyond the endpoints a and b. At these new meshpoints, new knots δ need to be introduced as well. Assuming that exactly μ new meshpoints are placed at each endpoint, then the new meshpoint sequence is $x_{-\mu}, ..., a = x_0, ...x_M = b, ..., x_{M+\mu}$.

Now collocation is applied to the temporal domain $[0,T]$. Partition $\Pi^t := \{\Pi^t_N : 0 = t_0 < ... < t_n = T\}$ and take the functions $w_j(t)$, $j = 1, ..., M\delta$ in a polynomial spline space. Introduce the temporal collocation points $\hat{T}_N := \{\hat{t}_n := t_n + \gamma_1 \tau_n : \gamma_1 \in [0,1],\ 0 \leq n \leq N-1\}$. Letting $\tau_n := t_{n+1} - t_n$ for $n = 0, ..., N-1$, we have

$$\begin{aligned} \frac{d\mathbf{w}}{dt} &= -\frac{1}{\tau_n}(\mathbf{b}^{(n)} - \mathbf{a}^{(n)}), \\ \mathbf{w} &= (1-\gamma_1)\mathbf{a}^{(n)} + \gamma_1 \mathbf{b}^{(n)}, \end{aligned}$$

on (t_n, t_{n+1}).

Applying this to the KdV, we have

$$\begin{aligned} \mathbf{B}\mathbf{w}' - 6\mathbf{B}\mathbf{w} \otimes \mathbf{B}'\mathbf{w} + \mathbf{B}'''\mathbf{w} &= 0, \\ \mathbf{B}\mathbf{w} &= \Phi \quad \text{initial condition.} \end{aligned}$$

Using a piecewise linear time interpolant, the initial condition becomes

$$\mathbf{B}(\mathbf{a}^{(0)}(1-\gamma_1)+\mathbf{b}^{(0)}\gamma_1)=\mathbf{B}\mathbf{a}^{(0)}=\Phi.$$

The algorithm is as follows. First, solve the equation directly above for $\mathbf{a}^{(0)}$. Next, for a given $\mathbf{a}^{(n)}$:

- Solve $(\frac{d\mathbf{w}}{dt}=(1-\gamma_1)\mathbf{a}^{(n)}+\gamma_1\mathbf{b}^{(n)})$ for $\mathbf{b}^{(n)}$ in order to get $\mathbf{w}(t)$ on the interval $[t_n,t_{n+1}]$.
- Solve for $U(x,t)$ anywhere on the strip $[a,b]\times[t_n,t_{n+1}]$.
- Obtain $\mathbf{a}^{(n+1)}$ from $\mathbf{b}^{(n)}$.

Using Gauss points at the mesh points in time yields superconvergent error estimates $O((\Delta t)^{2q},(\Delta x)^{2\delta})$ for q and δ Gauss points in each subinterval of time and space, respectively. Regarding stability of this method, Brunner and Roth state that "On a uniform mesh [...] and working in the spatial spline spaces $S(4,3,\Pi_M^x)$ and $S(5,3,\Pi_M^x)$, we found no evidence of instability with $q=1$ or $q=2$." [4, p. 379, 380]

3.5 Finite Element Methods

3.5.1 Petrov-Galerkin Method

Sanz-Serna and Christie used a Petrov-Galerkin ("PG") method to evaluate the KdV_+ equation [25]. The reader could recall that the Petrov-Galerkin method is similar to the Galerkin method with the exception of the allowance for the basis functions of the test and trial functions to differ. Sanz-Serna and Christie's method used the typical hat functions for the trial function [the u] and Hermite cubic polynomials for the test function [the v]. These authors state: "The Petrov-Galerkin approach enables us to use a C^0 interpolant, resulting in a much lower computational effort than that associated with the standard Galerkin method based on splines or Hermite cubics."

Starting from the KdV_+ equation, multiply by a test function v and integrate the dispersion term by parts twice.

$$\begin{aligned}
u_t + uu_x + \varepsilon u_{xxx} &= 0, \\
u_t v + uu_x v + \varepsilon u_{xxx} v &= 0, \\
\int_{-\infty}^{\infty} u_t v \, dx + \int_{-\infty}^{\infty} uu_x v \, dx + \varepsilon \left(u_{xx} v|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_{xx} v_x \, dx \right) &= 0, \\
(u_t, v) + (uu_x, v) + \varepsilon \left(u_{xx} v|_{-\infty}^{\infty} - u_x v_x|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u_x v_{xx} \, dx \right) &= 0, \\
(u_t, v) + (uu_x, v) + \varepsilon \left(u_{xx} v|_{-\infty}^{\infty} - u_x v_x|_{-\infty}^{\infty} + (u_x, v_{xx}) \right) &= 0,
\end{aligned}$$

where $(\cdot, \cdot)$ denotes the L_2 inner product: $(f, g) = \int_{-\infty}^{\infty} f(x) g(x) \, dx$.

Above, we have implicitly demanded that $v(x) \in C^1$. Pretending that we have taken v as a hostage, we further demand that $v(x) \to 0$ as $|x| \to \infty$ and we retrieve that

$$(u_t, v) + (uu_x, v) + \varepsilon (u_x, v_{xx}) = 0. \tag{3.7}$$

Using the finite element method, we introduce the uniformly-spaced mesh $x_0 < x_1 < \ldots < x_n$ with spacing h and use finite elements in the spatial domain for both test and trial functions:

$$\begin{aligned}
U(x, t) &= \sum_{i=0}^{n} U_i(t) \phi_i(x), \\
v(x, t) &= \sum_{i=0}^{n} v_j(t) \psi_j(x)
\end{aligned}$$

and demand that the trial functions ϕ_i have compact support. The capital U denotes the approximate solution. Equation (3.8) then becomes

$$\begin{aligned}
(U_t, v_j \psi_j) + (UU_x, v_j \psi_j) + \varepsilon (U_x, (v_j \psi_j)_{xx}) &= 0, \\
(U_t, v_j \psi_j) + (UU_x, v_j \psi_j) + \varepsilon (U_x, v_j (\psi_j)_{xx}) &= 0, \\
v_j (U_t, \psi_j) + v_j (UU_x, \psi_j) + \varepsilon v_j (U_x, (\psi_j)_{xx}) &= 0, \\
(U_t, \psi_j) + (UU_x, \psi_j) + \varepsilon (U_x, (\psi_j)_{xx}) &= 0,
\end{aligned}$$

where $j = 0 : n$.

Next, we choose the trial function $\phi(x)$ to be the familiar piecewise linear 'hat function' at each node x_i. Thus $\phi_i(x_i) = \delta_{ij}$, the more familiar Kronecker

delta function. Thus, $U_i(x_i,t) = U_i(t)\phi_i(x_i) = U_i(t)\delta_{ij} = U_i(t)$ when $i = j$. Next, we choose the test function $\psi(x)$. Since we are using a Petrov-Galerkin method rather than the Galerkin method, we are no longer forced to use the same function as both the test and the trial function. We define

$$\psi(x) = \psi\left(\frac{x - x_0}{h} - i\right).$$

For purposes of accuracy, we want a five-point approximant for u_{xxx}. Thus ψ must have support on $[-2,2]$, which means that it will be a cubic polynomial in each interval $[i, i+1]$ for $i = -2, -1, 0, 1$. Choosing Hermite polynomials, we have that

$$\sigma(x) = \begin{cases} (|x|-1)^2(2|x|+1) & \text{if } |x| \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\rho(x) = \begin{cases} x(|x|-1)^2 & \text{if } |x| \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

with

$$\begin{aligned} \sigma(0) = \rho'(0) &= 1, \\ \sigma(-1) = \sigma(1) &= 0, \\ \sigma'(-1) = \sigma'(0) = \sigma'(1) &= 0, \\ \rho(-1) = \rho(0) = \rho(1) &= 0, \\ \rho'(-1) = \rho'(1) &= 0. \end{aligned}$$

Finally,

$$\begin{aligned} \psi(x) = \alpha_{-1}\sigma(x+1) + \alpha_0\sigma(x) + \alpha_1\sigma(x-1) + \\ \beta_{-1}\rho(x+1) + \beta_0\rho(x) + \beta_1\rho(x-1), \end{aligned}$$

where $\alpha_i = \psi(i)$ and $\beta_i = \psi'(i)$.

With these at our disposal, the system $(U_t, \psi_j) + (UU_x, \psi_j) + \varepsilon(U_x, (\psi_j)_{xx}) =$

0 becomes (hold your breath)

$$\begin{aligned}
\tfrac{1}{60}\ & (9\alpha_1+2\beta_1)\dot{U}_{i+2}+(9\alpha_0+42\alpha_1+2\beta_0)\dot{U}_{i+1}\\
+\ & (42\alpha_0+9\alpha_1+9\alpha_{-1}-2\beta_1+2\beta_{-1})\dot{U}_i\\
+\ & (9\alpha_0+42\alpha_{-1}-2\beta_0)\dot{U}_{i-1}+(9\alpha_{-1}-2\beta_{-1})\dot{U}_{i-2})\\
+\ & \frac{1}{60h}((9\alpha_1+2\beta_1)U_{i+2}^2+(12\alpha_1+\beta_1)U_{i+1}U_{i+2}\\
+\ & (9\alpha_0+2\beta_0-6\beta_1)U_{i+1}^2+(12\alpha_0-12\alpha_1+\beta_0+\beta_1)U_iU_{i+1}\\
+\ & (9\alpha_{-1}-9\alpha_1-6\beta_0+2\beta_{-1}+2\beta_1)U_i^2\\
-\ & (12\alpha_0-12\alpha_{-1}-\beta_0-\beta_{-1}U_{i-1}U_i-(9\alpha_0-2\beta_0+6\beta_{-1})U_{i-1}^2\\
-\ & (12\alpha_{-1}-\beta_{-1})U_{i-2}U_{i-1}-(9\alpha_{-1}-2\beta_{-1})U_{i-2}^2)\\
+\ & \frac{\varepsilon}{h^3}(-\beta_1U_{i+2}+(2\beta_1-\beta_0)U_{i+1}+(2\beta_0-\beta_1-\beta_{-1})U_i\\
+\ & (2\beta_{-1}-\beta_0)U_{i-1}-\beta_{-1}U_{i-2})=0,
\end{aligned}\tag{3.8}$$

for $i=0:n$, where $\dot{U}$ denotes a partial derivative with respect to t. We set U which lie off of the mesh as zero; that is $U_{-2}=U_{-1}=U_{n+1}=U_{n+2}=0$.

By [25, p. 79], Taylor expansions to equation (3.8) yield the requirement that we set the relationship of the α and β as follows:

$$\begin{aligned}
\alpha_{-1}+\alpha_0+\alpha_1&=1,\\
\beta_{-1}+\beta_0+\beta_1&=0,\\
\beta_{-1}-\beta_1&=0.
\end{aligned}\tag{3.9}$$

We demand that the test functions be symmetric, so we have more constraints due to the conservation properties [25, p. 97]:

$$\begin{aligned}
\alpha_{-1}&=\alpha_1\\
\beta_{-1}&=-\beta_1\\
\beta_0&=0
\end{aligned}\tag{3.10}$$

From equations (3.9) and (3.10), we see that $-2\beta_1=1$, so $\beta_{-1}=\frac{1}{2}$, $\beta_1=-\frac{1}{2}$. With β_0, β_{-1}, β_1 as such, we can only choose one free parameter: α_1. From here we consider test functions $\psi(x)$ which depend on that single parameter $\alpha=\alpha_1$ [25, p. 97].

Thus, our PG method is as follows, broken down via each term:

$$\begin{aligned}
U_t=\frac{1}{60}((9\alpha-1)\dot{U}_{i+2}+(9+24\alpha)\dot{U}_{i+1}+(44-66\alpha)\dot{U}_i\\
+(9+24\alpha)\dot{U}_{i+1}+(9\alpha-1)\dot{U}_{i-2}),
\end{aligned}$$

$$UU_x = (1/120h)(18\alpha - 2)U_{i+2}^2 + (24\alpha - 1)U_{i+2}U_{i+1}$$
$$+(24 - 36\alpha)U_{i+1}^2 + (23 - 72\alpha)U_{i+1}U_i - (23 - 72\alpha)U_iU_{i-1}$$
$$-(24 - 36\alpha)U_{i-1}^2 - (24\alpha - 1)U_{i-1}U_{i-2} - (18\alpha - 2)U_{i-2}^2,$$

and

$$U_{xxx} = \frac{1}{2h^3}(U_{i+2} - 2U_{i+1} + 2U_{i-1} - U_{i-2}).$$

3.5.2 The 'modified' Petrov-Galerkin method

The modified Petrov-Galerkin method (mPG) for the KdV is simplistic in nature; in fact, it takes the same approximation approach as the method by Greig and Morris. The nonlinear term uu_x is written as $\left(\frac{u^2}{2}\right)_x$ and is approximated by

$$\frac{1}{48h}((12\alpha - 1)U_{i+2}^2 + (14 - 24\alpha)U_{i+1}^2 - (14 - 24\alpha)U_{i-1}^2 - (12\alpha - 1)U_{i-2}^2),$$

which makes the method fourth-order accurate [25].

The results were astounding.[11] As mentioned in the introduction, the mPG method was superior to the other methods used. Due to the fourth-order accuracy in space, the error in the mPG method shrank more quickly than the regular PG method.

3.5.3 Local Discontinuous Galerkin Method

In what might be the called the most significant [and interesting] numerical method development of the 21st century with respect to the KdV equation, Yan & Shu (2001) [28] use a combination of a discontinuous Galerkin

[11] Sanz-Serna and Christie used the initial condition $f(x) = 3c\ \mathrm{sech}^2(kx + d)$ with the parameters $c = 0.3$, $\varepsilon = 0.000484$, $\sqrt{c/4\varepsilon}$, $d = -k$, and $\alpha = \frac{1}{6}$.

finite elements and stable nonlinear high order Runge-Kutta methods. Respectively, these are applied to the spatial and temporal dimensions. First they write the equation as a first order system and apply the LDG method to it. As a result, three systems of first order arise, each of which is assigned a corresponding test function.

The method is stable for general nonlinearities of the form

$$u_t + f(u)_x + (r'(u)g(r(u)_x)_x)_x = 0.$$

Introduce the variables

$$\begin{aligned} q &= r(u)_x \\ p &= g(q)_x, \end{aligned}$$

and then the above equation can be written as

$$\begin{aligned} u_t + (f(u) + r'(u)p)_x &= 0, \\ p - g(q)_x &= 0, \\ q - r(u)_x &= 0. \end{aligned}$$

Discretizing with the discontinuous Galerkin method, we use three test functions v, w, z, integrate over the interval $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ $(j = 1 : N)$ and integrate by parts. We are looking for piecewise polynomials $u, p, q \in V_{\Delta x}$ such that

$$\begin{aligned} \int u_t v \, dx - \int (f(u) + r'(u)p)v_x \, dx + (\hat{f} + \hat{r'}\hat{p})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} \\ -(\hat{f} + \hat{r'}\hat{p})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = 0, \\ \int pw \, dx + \int g(q)w_x \, dx - \hat{g}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{g}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0, \\ \int qz \, dx + \int r(u)z_x \, dx - \hat{r}_{j+\frac{1}{2}} z^-_{j+\frac{1}{2}} + \hat{r}_{j-\frac{1}{2}} z^+_{j-\frac{1}{2}} = 0, \end{aligned}$$

where all $\int$ are over I_j. Definition of what we mean by the hatted functions (numerical fluxes) is necessary, and it is these choices that will ensure

stability:

$$\begin{aligned}
\hat{f} &= \hat{f}(u^-, u^+), \\
\hat{r'} &= \frac{r(u^+) - r(u^-)}{u^+ - u^-}, \\
\hat{p} &= p^+, \\
\hat{g} &= \hat{g}(q^-, q^+), \\
\hat{r} &= r(u^-),
\end{aligned}$$

where each hatted function $\hat{m}(u^-, u^+)$ is a monotone flux for $m(u)$. That is, $\hat{m}(u^-, u^+)$ is Lipschitz continuous in both arguments, has consistency such that $\hat{m}(u, u) = m(u)$, and is nonincreasing in u^+ and nondecreasing in u^-. Define the flux as the Lax-Friedrichs flux

$$\hat{f}(u^-, u^+) = \frac{1}{2}\left(f(u^-) + f(u^+) - \alpha(u^+ - u^-)\right),$$

where $\alpha = \max_u |f'(u)|$.

This scheme is L^2 stable:

$$\frac{d}{dt}\int_{I_j}\left(\frac{u^2(x,t)}{2}\right)dx + \left(\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}\right), \tag{3.11}$$

where the $\hat{H}$ are numerical entropy fluxes. This expression can be summed over all j to show L^2 stability [28, p. 776].

The error estimate for this scheme is

$$||u(x,t) - U(x,t)||_2 \leq C\Delta x^{k+\frac{1}{2}}.$$

This method with an implicit θ scheme,

$$\begin{aligned}
\frac{u^{n+1} - u^n}{\Delta t} &= R(u^{n+\theta}), \\
u^{n+\theta} &= (1-\theta)u^n + \theta u^{n+1},
\end{aligned}$$

where we impose the restriction $\frac{1}{2} \leq \theta \leq 1$, is stable as long as it satisfies a cell entropy inequality and the L^2 stability (3.11). The proof of this can be found upon [28, p. 777].

3.6 Summary of Numerical Methods

As mentioned before, the nonlinear term is sometimes written as $-6uu_x$, $+6uu_x$, or $+uu_x$. In the following table, the equation will be called KdV, +KdV, or KdV_+ respectively. Thus, +KdV will mean $u_t + 6uu_x + u_{xxx}$.

Table 1: Summary of Numerical Methods.

Equation	Spatial Domain	Name of Method	Reference
KdV_+	$\mathbb{R}$	Zabusky-Kruskal FDM	[29]
KdV_+	$\mathbb{R}$	Greig & Morris Hopscotch	[14]
KdV_+	$\mathbb{R}$	Goda's scheme	[13]
+KdV	$[0, 2\pi]$	Fornberg & Whitham Pseudospectral	[9]
+KdV	$\mathbb{R}$	proposed local scheme	[27]
KdV	$[0, 2\pi]$	semi-implicit scheme	[5]
KdV	$\mathbb{R}$	collocation in space and time	[4]
KdV_+	$\mathbb{R}$	Petrov-Galerkin method	[25]
KdV_+	$\mathbb{R}$	modified Petrov-Galerkin method	[25]
+KdV	$\mathbb{R}$	local discontinuous Galerkin method	[28]

4 Chosen Experimental Methods

The author of this paper chose the following methods to employ. Starred (∗) methods indicate that the author has included a modification to that method.

- Zabusky-Kruskal Finite Difference Method(∗)
- Pseudospectral Method
- Fornberg-Whitham Pseudospectral Method

The reader should note that for graphs which compare the analytical solution to the experimental solution, the former will be represented by a solid line, and the experimental solution will be represented by a dashed line.

4.1 Zabusky-Kruskal FDM

Similar to previously discussed, we apply the method of Zabusky-Kruskal to the +KdV, $u_t + 6uu_x + u_{xxx} = 0$. Assume a uniform mesh in the spatial domain with step size Δx.

The KdV equation is approximated just as before, with the addition of the coefficient +6 in front of the nonlinear term. The interim steps are omitted to evade the possibility of the reader conducting narcolepsy:

$$\begin{aligned} u_t &+ 6uu_x + u_{xxx} = 0, \\ u_j^{n+1} &= u_j^{n-1} - 2\frac{\Delta t}{\Delta x}(u_{j+1}^n + u_j^n + u_{j-1}^n)(u_{j+1}^n - u_{j-1}^n) \\ &- \frac{\Delta t}{\Delta x^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n). \end{aligned}$$

requires that

$$\frac{\Delta t}{\Delta x}\left|-2u_{max} + \frac{1}{(\Delta x)^2}\right| \leq \frac{2}{3\sqrt{3}}$$

in order to be stable.

For simplicity, both the initial condition $u(x,0)$ and the first step $u(x,\Delta t)$ were given to the program (a copy of which is located in the Appendices).

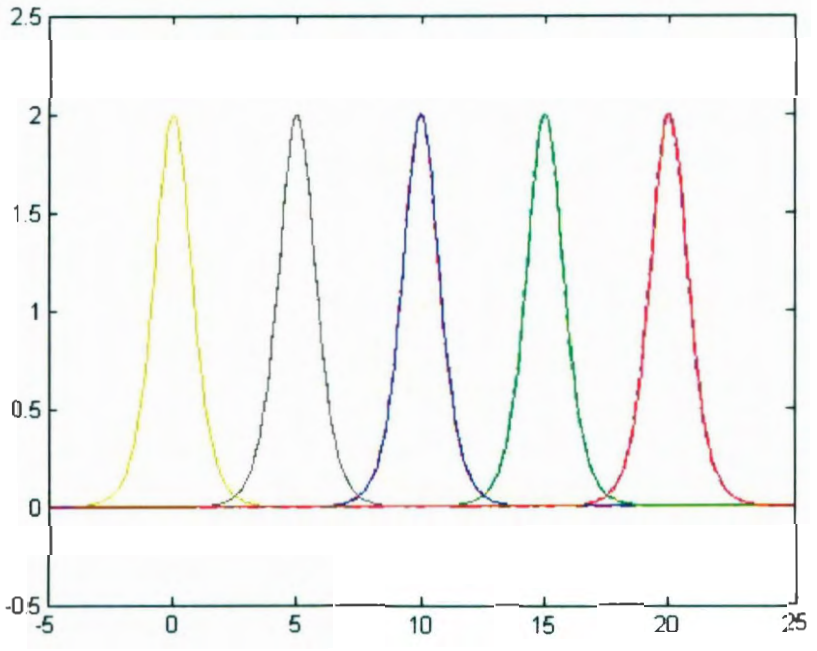

Figure 2: ZK Finite Difference method with $N = 2^9$, $\Delta t = 3.8641e-004$, plotted at $2\Delta t, \frac{1}{4}T, \frac{1}{2}T, \frac{3}{1}T, T$ where $T = 5.0$. The error of the method for these parameters is 0.0881.

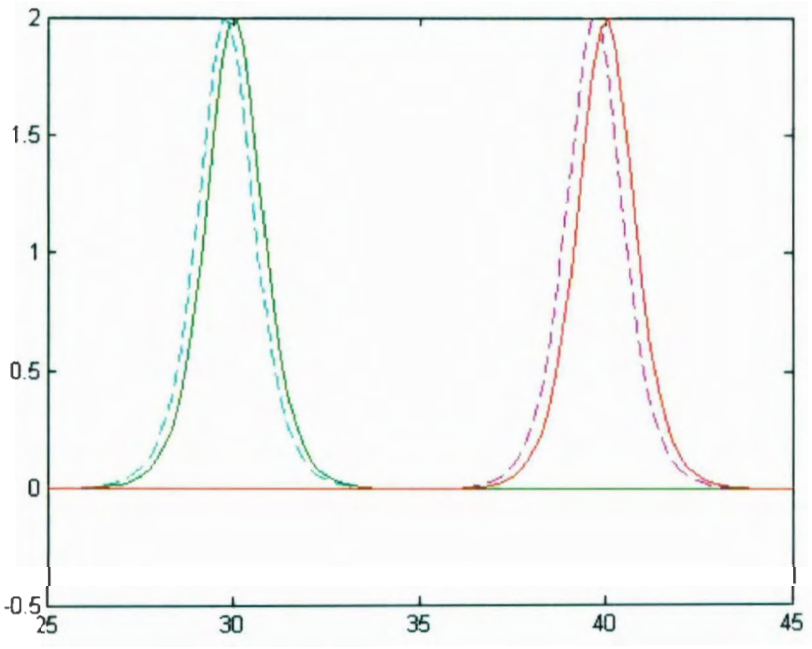

Figure 3: ZK Finite Difference method with $N = 2^9$, $\Delta t = 3.8641e-004$, plotted at $\frac{3}{4}T, T$ where $T = 10.0$. The error of the method for these parameters is 0.4661.

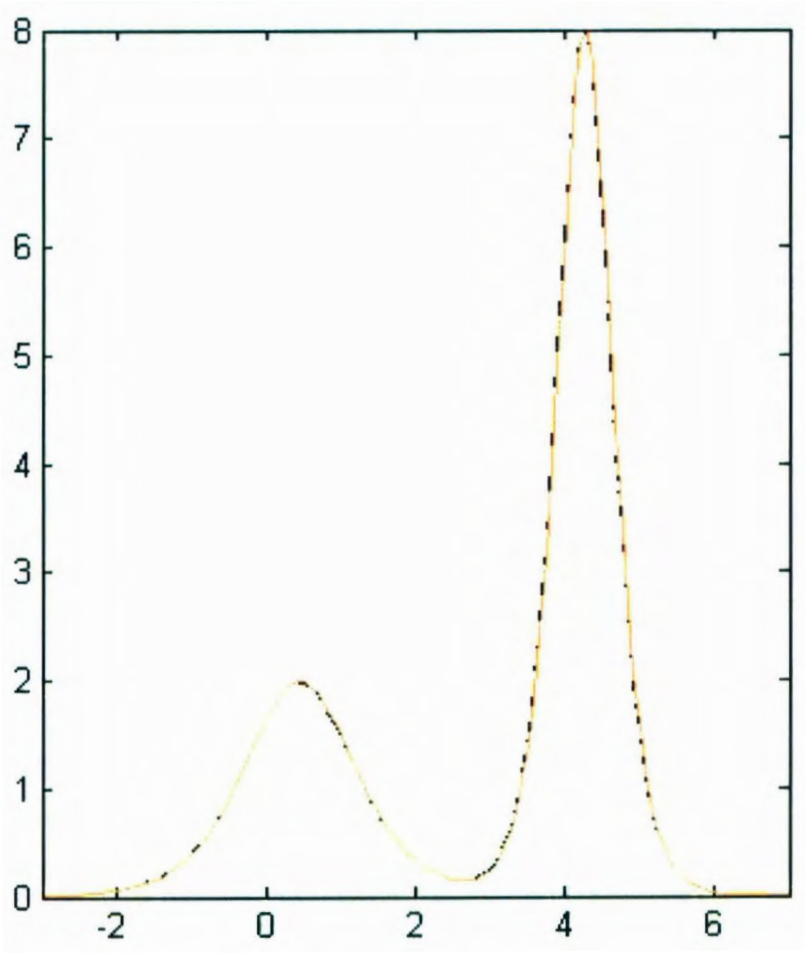

Figure 4: ZK Finite Difference method with $N = 2^9$, $\Delta t = 3.8641e{-}004$, plotted at T where $T = 0.25$, after a large (unshown) soliton splits into two smaller solitons. The error of the method for these parameters is 0.1769.

4.2 mZK : modified Zabusky-Kruskal

Consider the central difference approximation to the nonlinear term $+6uu_x$. Instead of an average over three points and a central difference approximation to the first derivative, we derive an approximation in a different way. Observe that

$$6uu_x = 3(u^2)_x \tag{4.1}$$

Now we approximate $3(u^2)_x$ using a central difference approximation. Define $w(x,t) := u^2(x,t)$. Then we have

$$\begin{aligned} 3(u^2)_x &= 3w_x \\ &\approx \frac{3}{2\Delta x}(w^n_{j+1} - w^n_{j-1}) \\ &= \frac{3}{2\Delta x}((u^n_{j+1})^2 - (u^n_{j-1})^2) \end{aligned}$$

which has a truncation error of $O(\Delta x^2)$, that is, exactly

$$\Delta x^2 \, |3u_x u_{xx} + uu_{xxx}|.$$

The mZK method is then

$$u^{n+1}_j = u^{n-1}_j - \frac{3\Delta t}{\Delta x}((u^n_{j+1})^2 - (u^n_{j-1})^2) - \frac{\Delta t}{\Delta x^3}(u^n_{j+2} - 2u^n_{j+1} + 2u^n_{j-1} - u^n_{j-2})$$

The result is refreshing. The method was run with the same inputs as the ZK method: $N = 2^{-9}$ and $\Delta t = 3.8641e-004$ for time $T = 5.0$. At time T, the method had an error of 0.0503 – but the error of the original ZK method was 0.0881 for the same inputs. This result represents a 43% increase in accuracy compared to the Zabusky-Kruskal finite difference method.

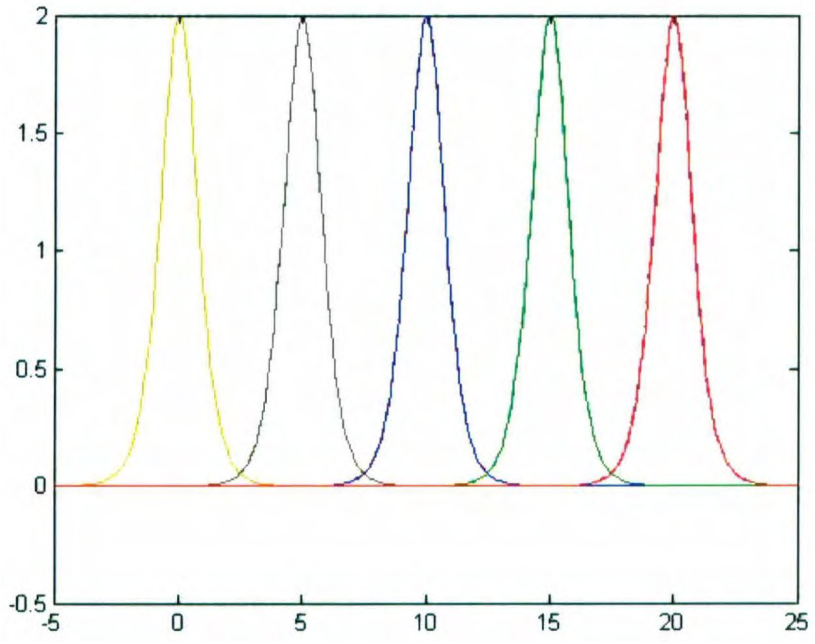

Figure 5: mZK Finite Difference method with $N = 2^9$, $\Delta t = 3.8641e-004$, plotted at $2\Delta t$, $\frac{1}{4}T$, $\frac{1}{2}T$, $\frac{3}{4}T$, T where $T = 5.0$. The error of the method for these parameters is 0.0503.

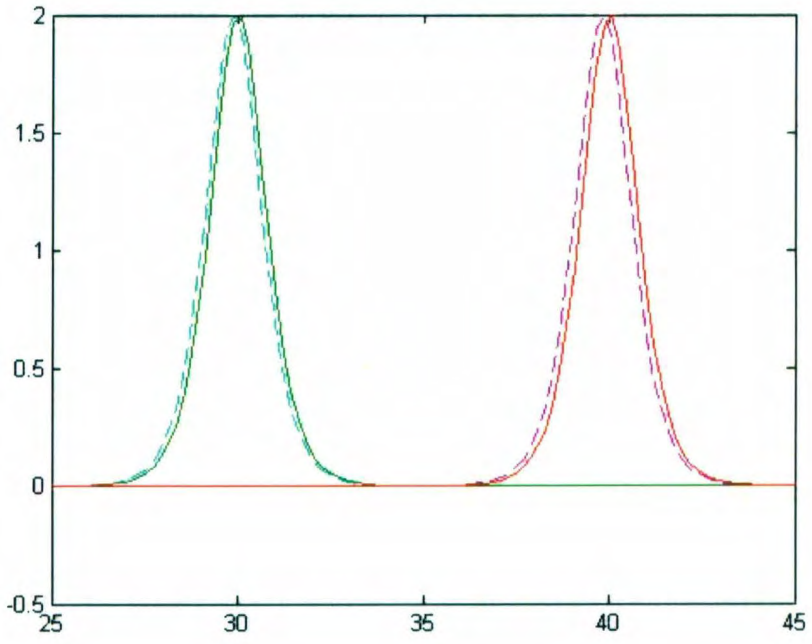

Figure 6: mZK Finite Difference method with $N = 2^9$, $\Delta t = 3.8641e-004$, plotted at $\frac{3}{4}T, T$ where $T = 10.0$. The error of the method for these parameters is 0.2717.

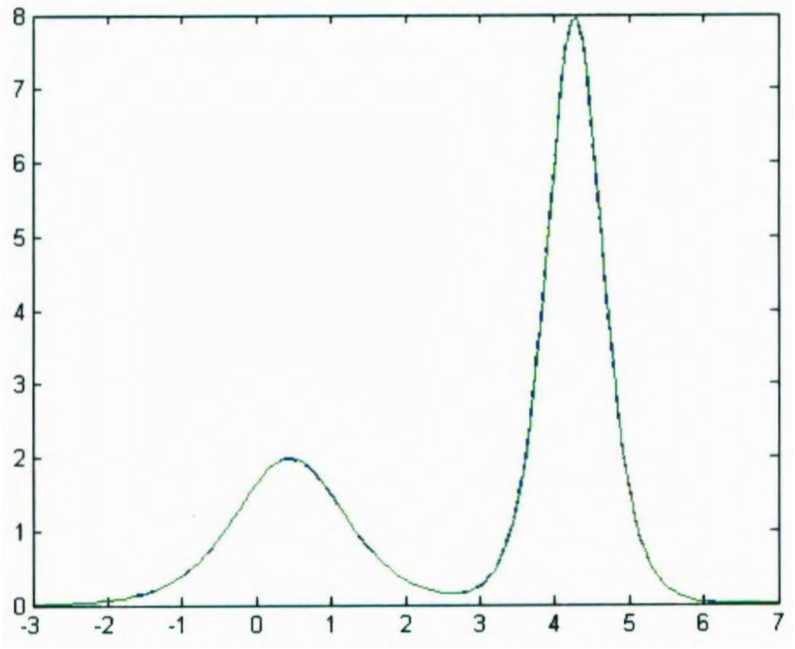

Figure 7: mZK Finite Difference method with $N = 2^9$, $\Delta t = 3.8641e-004$, plotted at T where $T = 0.25$. The error of the method for these parameters is 0.1443.

4.3 Pseudospectral Method and Fornberg-Whitham Pseudospectral Method

To demonstrate the pseudospectral method employed by Fornberg & Whitham, we will first develop the code which uses a pseudospectral method. Starting from the +KdV which has been normalized to $[0, 2\pi]$ (3.5), we have

$$u_t + 6\frac{\pi}{p}uu_x + \frac{\pi^3}{p^2}u_{xxx} = 0,$$
$$u(0,t) = u(2\pi,t).$$

As covered in §3.3.1, our method is

$$u_j^{n+1} = u_j^{n-1} - 12iu\Delta t\frac{\pi}{p}F^{-1}(kF(u)) + 2i\Delta t\frac{\pi^3}{p^3}F^{-1}(k^3F(u)) \qquad (4.2)$$

[see (3.6) and the appendix], where the first u_j^{n-1} will be analytically determined.

Inputs are $\Delta t = \frac{(\Delta x)^3}{\pi^3} \approx 0.0323(\Delta x)^3$, $T = 1.0$, the initial condition $u(x_j, 0) = 2\ \mathrm{sech}^2(x_j)$, the analytically determined $u(x_j, \Delta t) = 2\ \mathrm{sech}^2(x_j - 4\Delta t)$ and $N = 2^7$, the number of [equally spaced] points.[12] We compare the error of the experimental $U(x,t)$ with the exact solution at $T = 1.0$, which is $u(x_j, 1.0) = 2\ \mathrm{sech}^2(x_j - 4.0)$. The program yields an error of $6.1 * 10^{-3}$. The error was calculated by $||U(x_j, T) - u(x_j, T)||_\infty$ for all j..

The solutions U and u were plotted against each other. The experimental function U is represented by a dashed line, and the true solution u is represented by an unbroken lines. The solution is plotted when the program is at $U(x_j, 2\Delta t)$ (the first estimation) and the times $\frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T, T$ where T is the time to which the solution is calculated.

[12]The algorithm requires two timesteps of initial data, so we also input the solution at $t = \Delta t$, that is, $u(x_j, \Delta t) = U(x_j, \Delta t) = 2\ \mathrm{sech}^2(x_j - \delta t)$. Note that we could use a forward Euler for computation of $U(x_j, \Delta t)$.

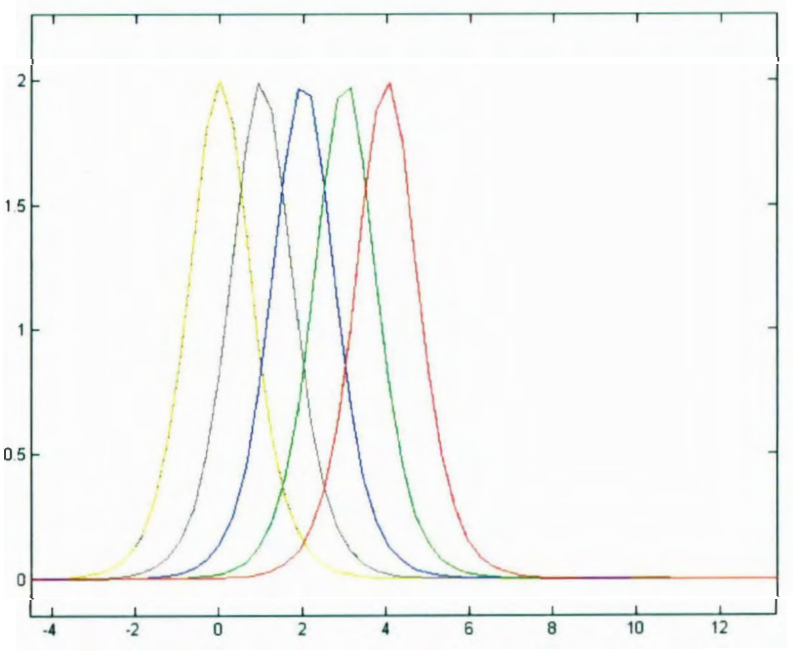

Figure 8: Pseudospectral method with $N = 2^7$, plotted at $\frac{1}{4}T, \frac{1}{2}T, \frac{3}{1}T, T$ where $T = 1.0$.

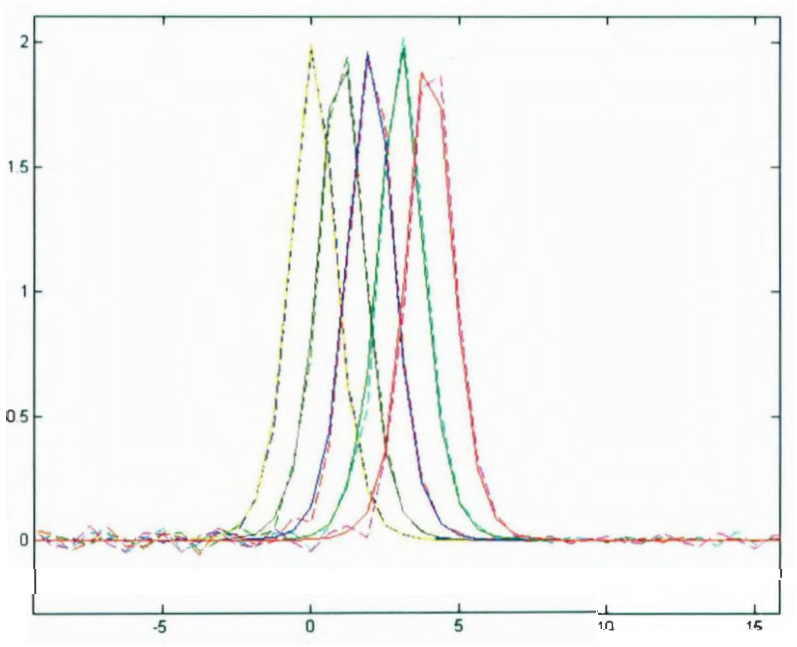

Figure 9: Due to the accuracy of the pseudospectral method, errors at this scale are nearly indistinguishable by the naked eye. In this graph, we have reduced the number of points to $N = 2^6$ to show that the method is less accurate with fewer points, and so that it is clear that both the experimental and true solutions are being graphed!

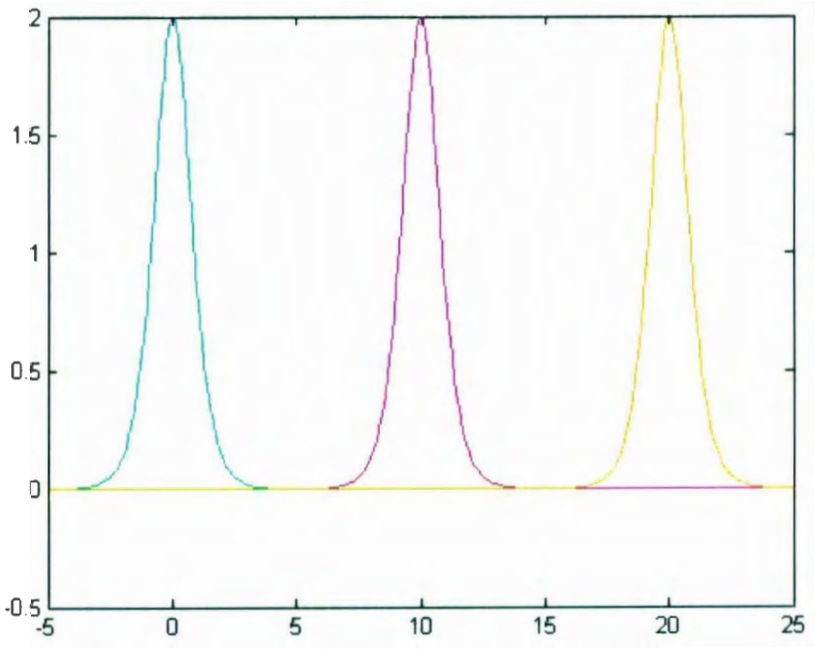

Figure 10: The pseudospectral method with $N = 2^9$, $T = 5.0$, $\Delta t = 5.1903e-005$. The solution is plotted for $t = 2\Delta t, \frac{1}{2}T, T$. The error at T is $3.1994e-004$.

With such a small error, it is clear that the program can reasonably estimate the true solution u for a single soliton solution. But what about a two soliton solution, and an interaction between two solitons moving at different speeds?

The program was again run with a change in the initial condition, which was replaced[13] with

$$u(x,t) = -12\frac{3 + 4\cosh(2x) + \cosh(4x)}{(3\cosh(x) + \cosh(3x))^2}.$$

Figure 11 shows the solitons splitting. Graphed are both U and u for the times $t = \frac{1}{2}T, T$ With $N = 2^8$, the method produces an error of $2.51 * 10^{-2}$.

[13]We also gave the program the first time step: $U(x,\Delta t) = -12\frac{3+4\cosh(2x-8*\Delta t)+\cosh(4x-64\Delta t)}{(3\cosh(x-28\Delta t)+\cosh(3x-36\Delta t))^2}$

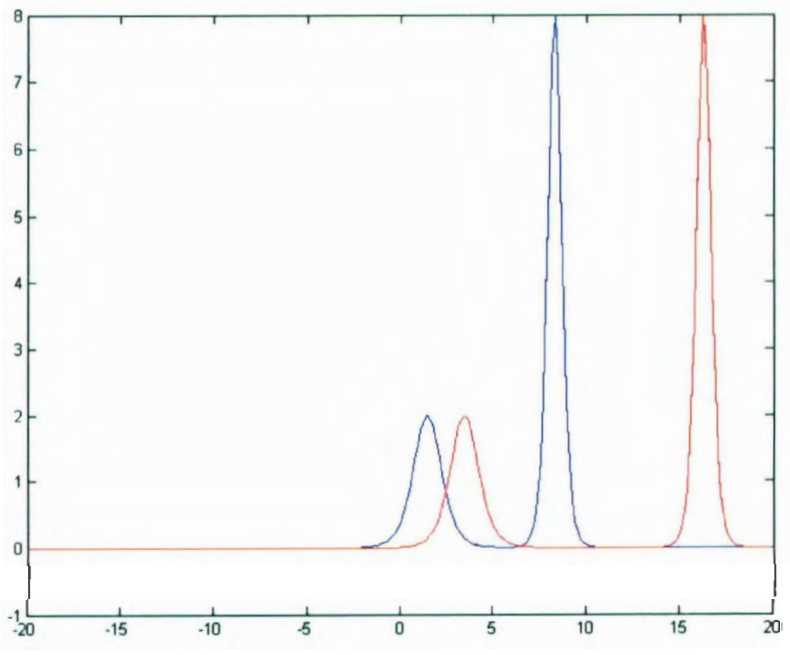

Figure 11: U and u are observationally similar.

Figure 12 plots two solitons interacting and subsequently splitting. Also seen is how the larger wave briefly "transfers" some of its energy (altitude) to the slower wave.

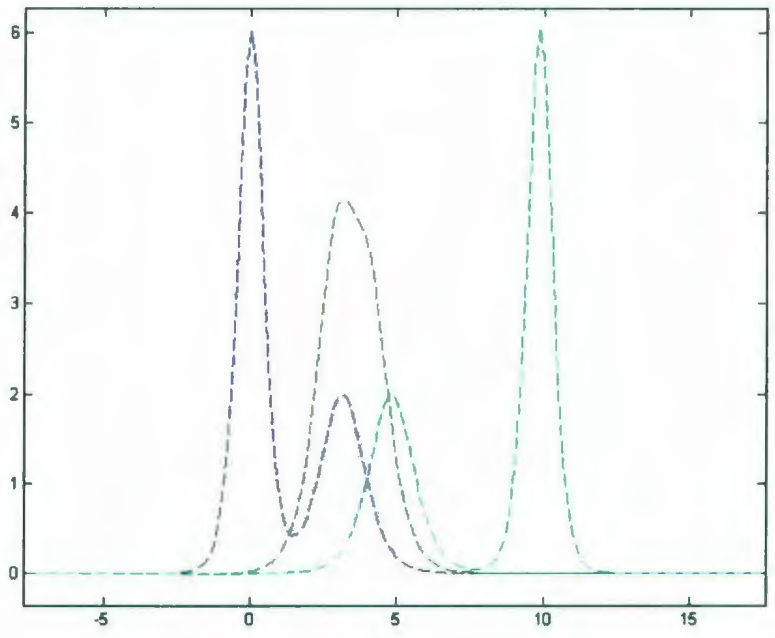

Figure 12: The larger soliton overtakes, interacts, and releases the smaller soliton.

Having an experimentally-functioning pseudospectral method, we implement a change in the approximation for u_{xxx}, in the footsteps of Fornberg & Whitham:

$$-2i\Delta t\frac{\pi^3}{p^3}F^{-1}(k^3F(u)) \longrightarrow -2iF^{-1}\left\{sin\left(\frac{\pi^3k^3}{p^3}\Delta t\right)F(u)\right\}$$

Implementing the change, we retrieve errors of $6.2*10^{-3}$ (for $N=2^7$) for the single soliton case and $2.80*10^{-2}$ (for $N=2^8$) in the two soliton case. As noted in [27], this implementation is slightly faster than (4.2).

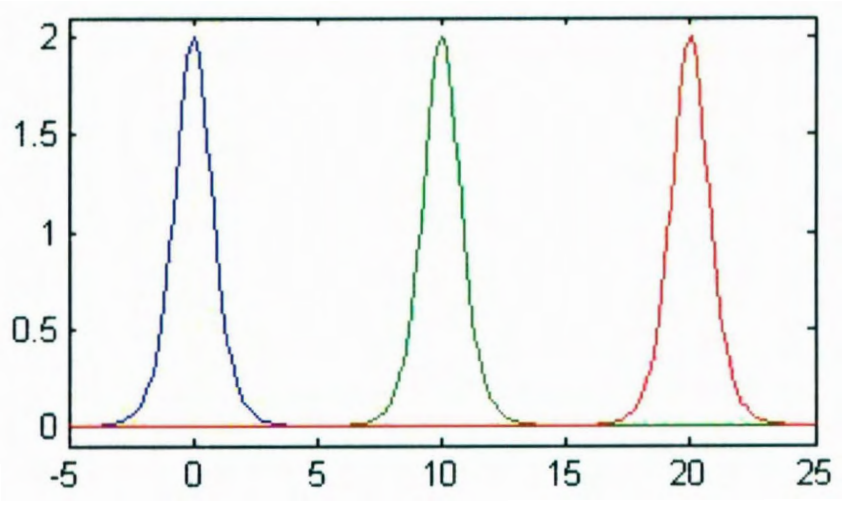

Figure 13: The FW pseudospectral method with $N = 2^9$, $T = 5.0$, $\Delta t = 2.3459e-004$. The solution is plotted for $t = 2\Delta t, \frac{1}{2}T, T$. The error at T is 0.0015.

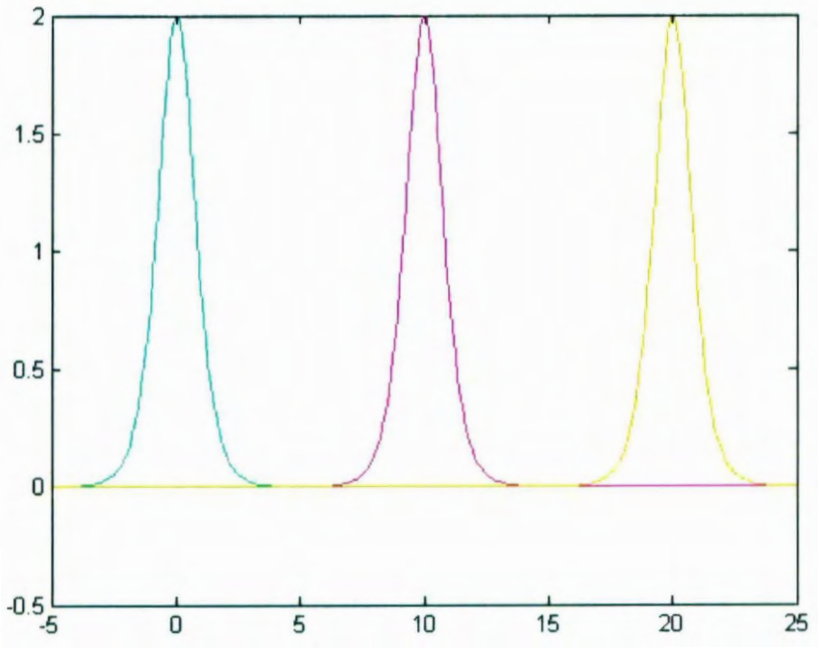

Figure 14: The FW pseudospectral method with $N = 2^9$, $T = 5.0$, $\Delta t = 5.1903e-005$. The Δt has been changed to the value $\Delta x^3/\pi^3$, the same as the unmodified pseudospectral method. The solution is plotted for $t = 2\Delta t, \frac{1}{2}T, T$. The error at T is $3.2169e-004$, which is less than the error for the unmodified pseudospectral method for the same Δt.

5 Outlook

5.1 KdV "Family" of equations

The KdV equations as earlier presented is not the only 'version' of the KdV equation. We examine some of them now.

5.1.1 Modified Korteweg-de Vries Equation

We first consider the *modified Korteweg-de Vries equation*[14]:

$$\frac{\partial u}{\partial t} - u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{5.1}$$

which is accomplished by using the a transformation in the KdV. This transformation, $\alpha u = \pm(6\beta)^{1/2}u_x - \beta u^2$, was discovered by Miura in 1968 [20].

5.1.2 Generalized Korteweg-de Vries Equation

The *generalized Korteweg-de Vries equation* is as follows:

$$\frac{\partial u}{\partial t} - u^p\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{5.2}$$

where p is an integer such that $p > 2$.

As if there were not enough "versions" of the KdV, there is another KdV family member: common literature refers to the *"Critical" General Korteweg-de Vries equation* (CgKdV). This is the case when the gKdV has $p = 5$. It is called the critical general KdV equation because the value $p = 5$ is the

[14]Gibbon: "[The mKdV equation] was named with less imagination [than the KdV equation]." [12]

critical point at which the solutions may blowup in finite time. That is, the solitary wave solutions of the gKdV equation are stable if and only if the aforementioned p is such that $p < 4$ [1]. Blowup of solutions in finite time may occur if $p > 5$ [19]. The existence of blowup in finite time when $p \geq 5$ in the Sobolev space H^1 is still an open problem.

These equations have been solved analytically for various function spaces, including the points in time where the solution experiences blowup.

5.2 Two-Dimensional KdV Equation

The KdV equation in two dimensions is called *Kadomtsev-Petviashvili equation* (hereafter *KP equation*) and is written

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

and has a large set of exact quasiperiodic solutions, each of which consist of N independent phases. Some 'real-life' applications of the KP equation have been observed, such as in the Strait of Gibraltar and the Dordogne River in southwestern France. One such observation was made in 1984 by the NASA space shuttle STS 41-G.

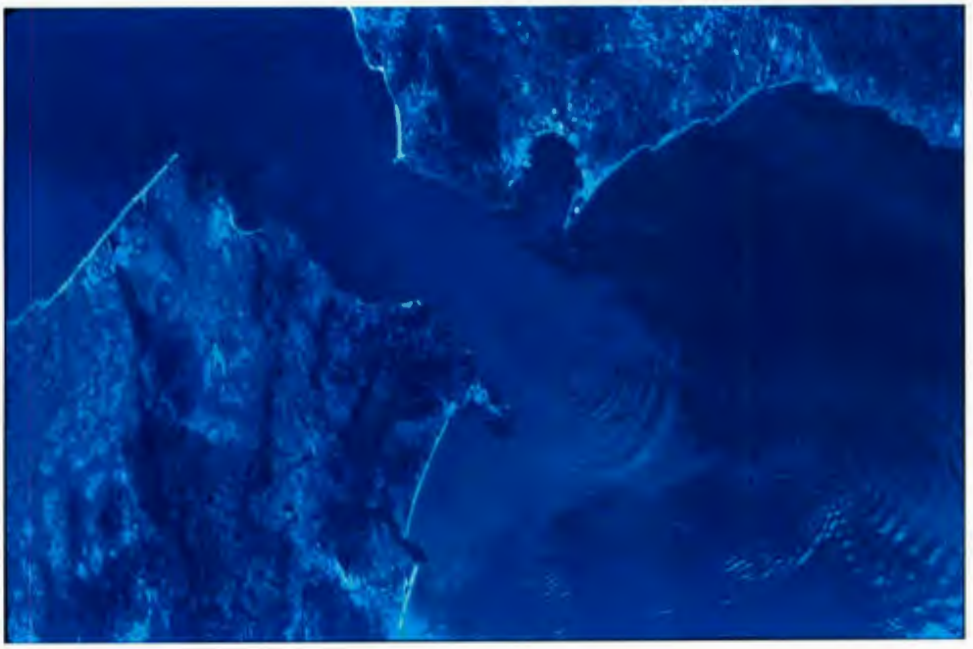

Figure 15: Picture taken by the STS 41-G in October 1984 [30]

5.3 Some Future Work

There remains more analysis and modifications to be done to the discussed numerical methods. Using the conservation form of the nonlinear term uu_x, that is, $\frac{1}{2}(u^2)_x$, perhaps more accurate methods can be devised, such as was done from the ZK to the mZK.

In addition, the conservation form of the nonlinear term could be used to modify other members of the KdV family of equations (see previous section), for instance when the nonlinear term is u^2u_x, as well as higher order terms found in the gKdV:

$$(u^p u)_x = \frac{1}{p+1}(u^{p+1})_x.$$

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A MATLAB Code Examples

A.1 MATLAB Code for Zabusky-Kruskal Finite Difference Method

See also [29], [27].

```
N = 2^9;
T = 10.0;
p = 100;
h = 2*p/N;
j = N;
J = [0:1:j-1];
x = 2 * pi * J / N;
x
for ii = 1:j
    xx(ii) = p * (x(ii)/pi - 1);
end
% Set initial condition.
for i = 1:j
    u1(i) = 2.*(sech(xx(i))).^2;
end
dt = h^3 / (4+6*(h^2*max(u1)));
for i = 1:j
    u2(i) = 2.*(sech(xx(i) - 4. *dt)).^2;
end
time = dt;
counter = 0;
b2 = -dt/(h) * 2;
v2 = dt/(h^3);
while time<(T)
    time = time + dt
    u(1) = u1(1)   + b2*(u2(2) + u2(1)   + u2(j))  *(u2(2) - u2(j))
                   - v2 *(u2(3) - 2*u2(2) + 2*u2(j)  - u2(j-1));
    u(2) = u1(2)   + b2*(u2(3) + u2(2)   + u2(1))  *(u2(3) - u2(1))
                   - v2 *(u2(4) - 2*u2(3) + 2*u2(1)  - u2(j));
    u(j-1)=u1(j-1) + b2*(u2(j) + u2(j-1) + u2(j-2))*(u2(j) - u2(j-2))
                   - v2 *(u2(1) - 2*u2(j) + 2*u2(j-2)- u2(j-3));
    u(j) = u1(j)   + b2*(u2(1) + u2(j)   + u2(j-1))*(u2(1) - u2(j-1))
                   - v2 *(u2(2) - 2*u2(1) + 2*u2(j-1)- u2(j-2));
    for i = 3:j-2
        u(i) = u1(i) + b2*(u2(i+1) + u2(i) + u2(i-1))*(u2(i+1) - u2(i-1))
```

```
                - v2 *(u2(i+2) - 2*u2(i+1) + 2*u2(i-1) - u2(i-2));
    end
    counter = counter + 1
    u1 = u2;
    u2 = u;
end
u_ans = 2.*(sech(xx - 4*time)).^2;
time
error = max(abs(u_ans - u))
plot(xx,u, '--',xx, u_ans, '-')
```

A.2 MATLAB Code for 'modified' Zabusky-Kruskal Finite Difference Method

```
N = 2^9;
T = 10.0;
p = 100;
h = 2*p/N;
j = N;
J = [0:1:j-1];
x = 2 * pi * J / N;
for ii = 1:j
    xx(ii) = p * (x(ii)/pi - 1);
end
% Set initial condition.
for i = 1:j
    u1(i) = 2.*(sech(xx(i))).^2;
end
dt = h^3 / (4+6*(h^2*max(u1)));
for i = 1:j
    u2(i) = 2.*(sech(xx(i) - 4. *dt)).^2;
end
time = dt;
counter = 0;
b2 = -dt/(h) * 3;
v2 = dt/(h^3);
while time<(T)
    time = time + dt
    u(1) = u1(1)   + b2* ((u2(2)).^2 - (u2(j)).^2)
                   - v2 *(u2(3) - 2*u2(2) + 2*u2(j)  - u2(j-1));
    u(2) = u1(2)   + b2* ((u2(3)).^2 - (u2(1)).^2)
                   - v2 *(u2(4) - 2*u2(3) + 2*u2(1)  - u2(j));
    u(j-1)=u1(j-1) + b2* ((u2(j)).^2 - (u2(j-2)).^2)
                   - v2 *(u2(1) - 2*u2(j) + 2*u2(j-2)- u2(j-3));
    u(j) = u1(j)   + b2* ((u2(1)).^2 - (u2(j-1)).^2)
                   - v2 *(u2(2) - 2*u2(1) + 2*u2(j-1)- u2(j-2));
    for i = 3:j-2
        u(i) = u1(i) + b2*((u2(i+1)).^2 - (u2(i-1)).^2)
                     - v2 *(u2(i+2) - 2*u2(i+1) + 2*u2(i-1) - u2(i-2));
    end
    counter = counter + 1
```

```
    u1 = u2;
    u2 = u;
end
u_ans = 2.*(sech(xx - 4*time)).^2;
time
error = max(abs(u_ans - u))
plot(xx,u, '--',xx, u_ans, '-')
```

A.3 MATLAB Code for Pseudospectral Method

See also [26].

```
clear,clc,close
N = 2^9;
T = 10.0;
p = 100;
norm = pi/p;
dx = 2*p/N ;
dt = (dx)^3 / (pi)^3; % 1/pi^3 \approx 0.0323

j = [0:1:N-1];
x = 2 * pi * j / N;

for ii = 1:N
    xx(ii) = p * (x(ii) /pi - 1);
end
for ii = 1:N
    u0(ii) = 2.*sech(xx(ii)).^2;
    u1(ii) = 2.*sech(xx(ii) - 4. * dt).^2;
end
counter = 0;
time = dt;
while time < T
  time
for ii = 1:N
U(ii) = (-1)^(ii-1)*u1(ii);
  end
U = fft(U);
for iii = 1:N
  FFT(iii) = (iii-1-N/2)*U(iii);
  FFT3(iii)=(iii-1-N/2)^3*U(iii);
  end
IFFT = ifft(FFT);
IFFT3= ifft(FFT3);
    i = sqrt(-1);
    for ii = 1:N
        first(ii) = (-1)^(ii-1) * -6*i* norm * u1(ii) * IFFT(ii);
        third(ii) = (-1)^(ii-1) * i * norm^3 * IFFT3(ii);
    end
```

```
    for ii = 1:N
        post(ii) = first(ii) + third(ii);
    end
    for ii = 1:N
        u(ii) = u0(ii) + 2 * dt * post(ii);
    end
    for ii = 1:N
        u0(ii) = u1(ii);
        u1(ii) = u(ii);
    end
    time = time + dt;
    counter = counter + 1
end
u_ans = 2.*sech(xx - 4*(dt * counter)).^2;
error = max(abs(u_ans - u1))
```

A.4 MATLAB Code for Fornberg-Whitham Pseudospectral Method

```
N = 2^9;
T = 10.0;
p = 100;
norm = pi/p;
dx = 2*p/N ;
dt = 3*(dx)^3 /(2 * (pi)^2) - 0.00001;
j = [0:1:N-1];
x = 2 * pi * j / N;

for ii = 1:N
    xx(ii) = p * (x(ii) /pi - 1);
end
for ii = 1:N
    u0(ii) = 2.*sech(xx(ii)).^2;
    u1(ii) = 2.*sech(xx(ii) - 4. * dt).^2;
end
counter = 0;
time = dt;
while time < T
  time
for ii = 1:N
U(ii) = (-1)^(ii-1)*u1(ii);
  end
U = fft(U);
for iii = 1:N
FFT(iii) = (iii-1-N/2)*U(iii);
FFT3(iii)=sin(norm^3 * (iii-1-N/2)^3 * dt)*U(iii);
  end
IFFT = ifft(FFT);
IFFT3= ifft(FFT3);
  i = sqrt(-1);
  for ii = 1:N
      first(ii) = (-1)^(ii-1) * -6 * i * norm * u1(ii) * IFFT(ii);
      third(ii) = (-1)^(ii-1) * i * IFFT3(ii);
  end
  for ii = 1:N
      u(ii) = u0(ii) + 2 * (dt * first(ii) + third(ii));
```

```
  end
  for ii = 1:N
      u0(ii) = u1(ii);
      u1(ii) = u(ii);
  end
  time = time + dt;
  counter = counter + 1
end

u_ans = 2.*sech(xx - 4*(dt * counter)).^2;
error = max(abs(u_ans - u1))
```

B Boussinesq's Paper

du camphre, pour remplacer celui qui avait été dissous ou entraîné par le pus : au bout d'une semaine, la plaie a commencé à marcher vers la cicatrisation, et les bourgeons charnus se sont produits de la façon la plus satisfaisante.

(Renvoi à la Commission des prix de Médecine et de Chirurgie.)

CORRESPONDANCE.

M. J.-F. Brandt, nommé Correspondant pour la Section d'Anatomie et de Zoologie, dans la séance du 4 juillet 1870, adresse de Saint-Pétersbourg ses remercîments à l'Académie.

M. Knoch adresse, de Saint-Pétersbourg, des remercîments pour la récompense dont ses travaux sur le Botriocéphale large ont été l'objet, dans le Concours de l'année 1869 (séance du 11 juillet 1870).

M. le Secrétaire perpétuel signale, parmi les pièces imprimées de la Correspondance :

1° Une brochure de *M. E. Favre*, portant pour titre : « Études sur la Géologie des Alpes. Le massif du Moléson et les montagnes environnantes dans le canton de Fribourg ».

2° Un volume de *M. Grimaud* (de Caux), portant pour titre : « L'Académie des Sciences pendant le siége de Paris, de septembre 1870 à février 1871 ».

HYDRODYNAMIQUE. — *Théorie de l'intumescence liquide appelée* onde solitaire *ou de translation, se propageant dans un canal rectangulaire.* Note de **M. J. Boussinesq**, présentée par M. de Saint-Venant.

« Je me propose d'établir théoriquement les lois des ondes observées par J. Scott-Russell et par M. Bazin dans des canaux rectangulaires, sensiblement horizontaux et de longueur indéfinie, contenant un liquide de profondeur constante, et aussi la formule que M. Bazin a déduite de ses expériences (*Savants étrangers*, t. XIX, et Rapport de M. Clapeyron, *Comptes rendus*, 10 août 1863), pour calculer la vitesse d'un courant de débit constant, propagé dans le même liquide. J'admettrai que celui-ci se trouve en repos au moment où les ondes l'atteignent : s'il était en mouvement par

Figure 16: Boussinesq's paper, page 1

filets rectilignes et parallèles de vitesses assez peu différentes, on pourrait, avec une certaine approximation, comme le prouvent de nombreuses expériences de M. Bazin, le supposer immobile par rapport à des axes coordonnés animés de la moyenne de ces vitesses, et rapporter les ondes à ce système d'axes.

» Les mouvements étudiés étant les mêmes sur toute la largeur du canal, il suffit de les considérer dans un plan vertical dirigé suivant sa longueur. Dans ce plan, je prendrai, suivant le sens de la propagation des ondes, le fond horizontal pour axe des x, et une verticale dirigée en haut pour axe des y; enfin j'appellerai H la profondeur constante du liquide en repos, y, ou $\mathrm{H} + h$ la profondeur dans les parties agitées, h, la valeur maximum de h, valeur dont le rapport à H sera néanmoins supposé assez petit; ρ la densité; u et v les composantes, à l'époque t, de la vitesse en (x, y); u_1 et v_1 les composantes pareilles en un point de la surface libre; enfin ω la vitesse de propagation des ondes.

» Je m'occuperai d'abord des ondes solitaires, dont les caractères distinctifs sont : 1° de produire, au moment de leur passage, des vitesses sensiblement constantes du fond à la surface, de manière que u et sa dérivée en x varient peu avec y; 2° de parcourir de grandes distances avec une vitesse de propagation constante et sans altération notable. Il suit de ce deuxième caractère que u, v sont seulement fonctions de $x - \omega t$, y, et aussi que les frottements sont insensibles et qu'on peut s'appuyer sur les équations ordinaires de l'hydrodynamique. Comme d'ailleurs u, v sont nuls autour de chaque molécule avant que l'onde y passe, ils s'y trouveront, à toute époque, d'après un théorème connu de Lagrange et de Cauchy, les dérivées partielles en x et y d'une fonction φ; et une formule usuelle donnera, p désignant l'excès de la pression en un point sur celle de l'atmosphère,

$$(1)\qquad \frac{p}{\rho} = g(\mathrm{H} - y) - \frac{d\varphi}{dt} - \frac{1}{2}\left(\frac{d\varphi^2}{dx^2} + \frac{d\varphi^2}{dy^2}\right) = g(\mathrm{H} - y) - \frac{d\varphi}{dt} - \frac{1}{2}(u^2 + v^2).$$

» Mais u et v n'étant fonctions que de $x - \omega t$, y, l'on a

$$(2)\qquad \frac{d^2\varphi}{dx\,dt} = -\omega\frac{d^2\varphi}{dx^2}, \quad \frac{d^2\varphi}{dy\,dt} = -\omega\frac{d^2\varphi}{dx\,dy}, \quad \text{d'où} \quad \frac{d\varphi}{dt} = -\omega\frac{d\varphi}{dx} + \text{fonct. arbitr. de } t.$$

Cette fonction arbitraire est nulle; car, pour $x = \infty$ et $y = \mathrm{H}$, p, u, v sont nuls; et, d'après (1), la dérivée de φ en t l'est également. Celle-ci peut donc être remplacée, dans (1), par $-\omega u$. Il résulte d'ailleurs de l'incompressibilité du liquide que le volume $\int_0^{y_1} u\,dy$, passé à travers une section normale

durant un instant δ, est égal à celui, $\omega\delta h$, qui se trouve de plus au delà de cette section au bout du même instant. A une première approximation, u ne dépend pas de y, et il vient successivement

$$(3)\qquad \begin{cases} u = \dfrac{\omega h}{\mathrm{H}}, \quad \dfrac{du}{dx} = \dfrac{\omega}{\mathrm{H}}\dfrac{dh}{dx}, \\ v = -\displaystyle\int_0^y \frac{du}{dx}\,dy = -\frac{\omega}{\mathrm{H}}\frac{dh}{dx}y, \quad \dfrac{du}{dy} = \dfrac{dv}{dx} = -\dfrac{\omega}{\mathrm{H}}\dfrac{d^2h}{dx^2}y. \end{cases}$$

« Multiplions la dernière (3) par dy et intégrons en déterminant la constante au moyen de la condition précédente d'incompressibilité, nous aurons

$$(4)\quad u = \frac{\omega h}{\mathrm{H}} - \frac{\omega}{6\mathrm{H}}\frac{d^2h}{dx^2}(\mathrm{H}^2 - 3y^2), \quad \text{d'où} \quad u_1 = \frac{\omega h}{\mathrm{H}}\left[1 - \frac{h}{\mathrm{H}} - \frac{\mathrm{H}^2}{3h}\frac{d^2h}{dx^2}\right].$$

Portant dans la relation (1), spécifiée pour la surface libre, les valeurs de v_1 et de u_1 données par (3) et (4), il vient

$$(5)\qquad \omega^2 = g\mathrm{H}\left[1 + \frac{3}{2}\frac{h}{\mathrm{H}} + \frac{\mathrm{H}}{2h}\frac{dh^2}{dx^2} + \frac{\mathrm{H}^2}{3h}\frac{d^2h}{dx^2}\right].$$

« L'intégrale première de cette équation, si l'on appelle C une constante, est

$$(6)\qquad \frac{dh^2}{dx^2} = -3\left(\frac{h}{\mathrm{H}}\right)^2 + \frac{3\omega^2}{g\mathrm{H}}\left(\frac{h}{3} - \frac{h}{\mathrm{H}}\right) + \mathrm{C}e^{-\frac{h}{\mathrm{H}}},$$

ou sensiblement, en déterminant C de manière que la dérivée de h en x s'annule pour $h = 0$, développant l'exponentielle jusqu'au terme en h^3, et observant qu'on peut, d'après (5), remplacer, dans ce terme, ω^2 par $g\mathrm{H}$,

$$(7)\qquad \frac{dh^2}{dx^2} = 3\left(\frac{\omega^2}{g} - \mathrm{H} - h\right)\frac{h^2}{\mathrm{H}^2}.$$

Au sommet de l'onde, où $h = h_1$, et où la dérivée de h en x est nulle, cette formule (7) devient $\omega^2 = g(\mathrm{H} + h_1)$, qui a été trouvée expérimentalement par J. Russell et vérifiée par M. Bazin. Si l'onde était négative ou que h_1 (alors valeur minimum de h) fût < 0, la même dérivée, nulle pour $h = h_1$, serait imaginaire, d'après (6), pour $h > h_1$: on ne peut donc pas appliquer aux ondes négatives la théorie actuelle, ni, par suite, l'hypothèse consistant à admettre que u, v dépendent seulement de $x - \omega t$, y, ou que, même à une seconde approximation, l'onde se propage uniformément et sans se déformer. En effet, J. Russell et M. Bazin ont reconnu que ces ondes s'altèrent promptement et qu'elles sont d'ailleurs suivies de plusieurs autres, alternativement positives et négatives : on doit se contenter jusqu'à présent, à

Figure 18: Boussinesq's paper, page 3

leur égard, de la première approximation, due à Lagrange et résumée par les deux formules $\omega^2 = gH$, $u\Pi = \omega h$.

« Si ωt désigne l'abscisse pour laquelle $h = h_0$, l'intégrale de (6) est

$$(8) \qquad \zeta h_0 = \left[2 + e^{\sqrt{\frac{3}{H^3}}(x-\omega t)} + e^{-\sqrt{\frac{3}{H^3}}(x-\omega t)} \right] h_0.$$

« La surface libre est donc symétrique par rapport à la verticale mobile $x = \omega t$, et est tout entière au-dessus du niveau $y = H$. Sa courbure, sensiblement mesurée par la dérivée seconde de h en x, a pour expression, d'après (5), le quotient par $2H^3$ de $3h(3h_0 - 3h)$: il y a deux points d'inflexion seulement, pour h égal aux deux tiers de h_0, et, par suite, une seule convexité ou onde formée par le liquide. Le volume fluide Q, qui constitue cette onde, est, par unité de largeur,

$$2\int_0^{h_0} h\frac{dx}{dh}\,dh = 4\sqrt{\frac{H^3 h_0}{3}},$$

intégrale dont la valeur s'obtient en substituant à la dérivée de x en h son expression tirée de (6) : on en déduit h_0 et ω en fonction de Q.

« Supposons actuellement que l'onde ne se termine pas à son arrière, comme il arrive si elle est produite par une effusion permanente de liquide ou par un refoulement continu de l'eau vers les x positifs. Les vitesses u, v ne pourront plus être partout de simples fonctions de $x - \omega t$, y; car, si l'onde se propageait d'après les lois précédentes, la surface, représentée par (8), finirait par s'abaisser, du côté des x négatifs, jusqu'au niveau $y = H$, où l'eau serait immobile, conséquence impossible dans l'hypothèse d'une onde sans fin. Les expériences de M. Bazin, tout en montrant l'uniformité du mouvement de propagation de la lame liquide, d'une hauteur constante h_0, qui forme le corps de l'onde, me paraissent établir, en effet, que les renflements placés à sa tête, et dont il appelle le premier et le plus élevé *onde initiale*, sont très-variables de forme et de hauteur : la théorie précédente ne s'applique donc plus. Toutefois, une démonstration donnée par M. de Saint-Venant (*Comptes rendus*, t. LXXI, p. 186, 18 juillet 1870) permet d'obtenir la formule expérimentale de M. Bazin,

$$\omega^2 = g(H + 1{,}5h_0),$$

même dans le cas où le rapport de h_0 à H ne serait pas très-petit. Cette démonstration, simple application du théorème sur les quantités de mouvement, consiste à considérer pendant un instant δ le volume liquide compris, au commencement de cet instant, entre deux sections $x = x_1$, où la profondeur est $H - h_0$, et $x = x_2$, où elle est H, et à égaler le produit

Figure 19: Boussinesq's paper, page 4

par θ de la différence des pressions $0{,}5\rho g(H + h_0)^2$, $0{,}5\rho g H^2$, exercées sur les deux sections, à la quantité de mouvement acquise par ce volume suivant les x, durant le même instant, quantité égale à $\rho\omega\theta(H + h_0)u$, en vertu de la progression de l'onde, moins celle, $\rho u\theta(H + h_0)u$, que possède le liquide, étranger au volume considéré, qui a traversé la section $x = x_0$ pendant l'instant θ. En remplaçant u par sa valeur tirée de la relation d'incompressibilité $(H + h_0)u = h_0\omega$, et négligeant un terme positif en h^2, généralement insensible et dont l'omission doit compenser d'ailleurs celle du frottement contre le fond et les bords, il vient bien

$$\omega^2 = g(H + 1{,}5h_0). \text{ »}$$

PHYSIQUE. — *Sur la dissociation au point de vue de la thermodynamique.* Note de **M. J. Moutier**, présentée par M. H. Sainte-Claire Deville.

« M. H. Sainte-Claire Deville, à qui l'on doit la découverte du phénomène de la dissociation, a montré que des gaz, réputés jusqu'alors indécomposables par la chaleur, éprouvent, au contraire, par l'effet de la chaleur, une décomposition progressive, caractérisée par une tension de dissociation susceptible d'être mesurée en millimètres de mercure, comme la tension d'une vapeur (1). Les expériences de M. Debray sur la dissociation du spath d'Islande et l'efflorescence des sels (2), celles de M. Isambert sur la dissociation de certains chlorures ammoniacaux à basse température (3), ont établi que la tension de dissociation croît avec la température, suivant une loi tellement régulière, que M. Lamy a pu faire servir la tension de dissociation à la mesure des températures (4).

« M. H. Sainte-Claire Deville a signalé le premier l'analogie qui existe entre la dissociation et la vaporisation, et tracé nettement la marche à suivre dans l'étude de la dissociation au point de vue de la thermodynamique. Lorsque la tension de dissociation est fonction de la température seule, comme cela paraît avoir lieu d'une manière générale pour tous les composés directs, les formules établies par M. Clausius (5) pour la vaporisation et la fusion sont directement applicables à la dissociation.

(1) *Leçons sur la dissociation* faites en 1864 devant la Société chimique; Paris, Hachette. — *Leçons sur l'affinité*, 1869, Hachette.

(2) *Comptes rendus*, t. LXIV, p. 603, et t. LXVI, p. 194.

(3) *Annales de l'École Normale*, t. V, p. 129.

(4) *Comptes rendus*, t. LXIX, p. 347, et t. LXX, p. 393.

(5) *Théorie mécanique de la chaleur*, traduite par F. Folie, t. I, p. 89, 41 et 404.

Figure 20: Boussinesq's paper, page 5